

## ANALYTIC TORSIONS ON CONTACT MANIFOLDS

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ABSTRACT. We propose a definition for analytic torsion of the contact complex on contact manifolds. We show it coincides with Ray–Singer torsion on any 3-dimensional CR Seifert manifold equipped with a unitary representation. In this particular case we compute it and relate it to dynamical properties of the Reeb flow. In fact the whole spectral torsion function we consider may be interpreted on CR Seifert manifolds as a purely dynamical function through Selberg-type trace formulae.

## 1. INTRODUCTION

As introduced by Ray and Singer in [33], the analytic torsion of a compact Riemannian manifold  $M$  may be seen as an infinite-dimensional analogue, on the de Rham complex  $(\Omega^*M, d)$ , of the Reidemeister–Franz torsion of a finite simplicial complex. More precisely, for  $\lambda \geq 0$ , let  $E_\lambda^k$  be the  $[0, \lambda]$ -spectral space of Hodge–de Rham Laplacian  $\Delta_k$  on  $k$ -forms. Then the cut-off subcomplex  $(E_\lambda^*, d)$  is finite-dimensional and its Reidemeister–Franz torsion satisfies

$$(1) \quad 2 \ln \tau_R(E_\lambda^*, d) = \ln \left( \prod_{k=0}^n \det(\Delta_k|_{E_\lambda^k})^{(-1)^{k+1}k} \right) = \sum_{k=0}^n (-1)^k k \zeta'(\Delta_k|_{E_\lambda^k})(0),$$

where  $\zeta(\Delta_k|_{E_\lambda^k})(s) = \text{Tr}(\Delta_k|_{E_\lambda^k}^{-s})$  is the truncated zeta function of  $\Delta_k$  on  $E_\lambda^k$ . Taking these  $(E_\lambda^*, d)$  as successive ‘approximations’ to the full de Rham complex, Ray–Singer defined the analytic torsion  $T_{RS}$  as being

$$(2) \quad T_{RS} = \exp \left( \frac{1}{2} \sum_{k=0}^n (-1)^k k \zeta'(\Delta_k)(0) \right),$$

while the Ray–Singer metric on  $\mathcal{L} = \det H^*(\Omega^*M, d)$  is given by

$$(3) \quad \| \cdot \|_{RS} = (T_{RS})^{-1} | \cdot |_{L^2(\Omega^*M)},$$

from the  $L^2$  metric induced on  $\mathcal{L}$  via identification of the cohomology by harmonic forms in  $\Omega^*M$ .

The first purpose of this work is to adapt this idea to the contact complex  $(\mathcal{E}^*, d_H)$ , a hypoelliptic differential form complex naturally defined ([36, 37]) on contact manifolds  $(M, H)$  of dimension  $2n+1$ . A specific feature of this complex is that the differential  $D = d_H : \mathcal{E}^n \rightarrow \mathcal{E}^{n+1}$

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*Date:* 4th February 2008.

*2000 Mathematics Subject Classification.* 58J52, 32V05, 32V20, 11M36, 37C30.

*Key words and phrases.* analytic torsion, contact complex, CR Seifert manifold.

First author supported in part by the French ANR-06-BLAN60154-01 grant. The second author gratefully acknowledges financial support received through the Japanese Government (MEXT) Scholarship for research students.

in ‘middle degree’ is a second-order operator, which is due to a slower spectral sequence convergence at this degree; see [38, Proposition 3.3]. In order to find, as above, finite-dimensional cut-off subcomplexes  $(E_\lambda^k, d_H)$  approximating  $(\mathcal{E}^*, d_H)$ , we are led to consider *fourth*-order Laplacians  $\Delta_k$  in *all* degrees  $k$ ; see (10). The Reidemeister–Franz torsion of each cut-off subcomplex is then easily written (see Proposition 2.8) as

$$4 \ln \tau_R(E_\lambda^*, d_H) = \ln \left( \prod_{k=0}^{2n+1} \det(\Delta_{k \lceil E_\lambda^k})^{(-1)^{k+1} w(k)} \right) = \sum_{k=0}^{2n+1} (-1)^k w(k) \zeta'(\Delta_{k \lceil E_\lambda^k})(0),$$

with  $w(k) = k$  for  $k \leq n$  and  $w(k) = k + 1$  for  $k > n$  being the natural contact-weight of forms in  $\mathcal{E}^k$ ; compare with (1). This leads us to define a candidate for the analytic torsion of the full contact complex by setting

$$T_C = \exp \left( \frac{1}{4} \sum_{k=0}^{2n+1} (-1)^{k+1} w(k) \zeta'(\Delta_k)(0) \right).$$

We define also a torsion function

$$\kappa(s) = \frac{1}{2} \sum_{k=0}^{2n+1} (-1)^{k+1} w(k) \zeta(\Delta_k)(s),$$

and a Ray–Singer metric  $\| \cdot \|_C$  on the determinant of the cohomology  $\det H^*(\mathcal{E}, d_H)$ ,

$$\| \cdot \|_C = T_C | \cdot |_{L^2(\mathcal{E})}.$$

(Our convention for  $T_C$  is inverse to Ray–Singer’s original definition (2), but is standard now since it is natural at the metric level, as compared to (3); see e.g. [13, 11].)

Having thus defined a torsion upon geometric and algebraic bases, we start then its analytical study. We first establish in Theorem 3.4 and Corollary 3.7 general variational formulae for  $\kappa(0)$ ,  $T_C = \exp(\kappa'(0)/2)$  and the Ray–Singer ‘contact’ metric  $\| \cdot \|_C$ . It turns out that  $\kappa(0)$  is a *contact* invariant given by the integration of an unknown universal polynomial in local curvature data. Its vanishing is necessary in order for the Ray–Singer metric to be even scale invariant under change of contact form  $\theta \mapsto K\theta$  for  $K$  constant. We do not know whether  $\kappa(0)$  vanishes in general except in dimension 3, as shown in Corollary 3.8. Therefore the rest of the paper deals with this lowest-dimensional case.

Corollary 3.8 also states that there exist CR-invariant and contact-invariant ‘corrections’ to this Ray–Singer metric. Namely there exist universal constants  $(C_i)_{1 \leq i \leq 4}$  such that, on any contact manifold of dimension 3,

$$\| \cdot \|_{\text{CR}} = \exp \left( C_1 \int_M R^2 \theta \wedge d\theta + C_2 \int_M |A|^2 \theta \wedge d\theta \right) \| \cdot \|_C,$$

is a CR-invariant (i.e. independent of contact form) metric on  $\det H^*(\mathcal{E}, d_H)$ , where  $R$  and  $A$  are the Tanaka–Webster scalar curvature and torsion. Moreover

$$\| \cdot \|_H^\nu = \exp(C_3 \nu(M)) \| \cdot \|_{\text{CR}} \quad \text{and} \quad \| \cdot \|_H^D = \exp(C_4 \bar{\eta}(D^*)) \| \cdot \|_{\text{CR}}$$

are contact-invariant metrics, where  $\nu(M)$  is the  $\nu$ -invariant of Biquard–Herzlich [6], and  $\bar{\eta}(D^*)$  is the CR-invariant correction to  $\eta(D^*)$ ; see [7, Theorem 9.4].

We do not know the values of the constants  $C_i$ . Though they are all related to the Heisenberg symbol of the Laplacians we consider, they might be difficult to compute: first since the  $\Delta_k$  are fourth-order, but also because the Heisenberg symbolic calculus ([4, 21], §3.1) suitable for the hypoelliptic contact complex, is highly non-commutative.

Our next purpose is to compare the two analytic torsions and metrics coming from the de Rham and contact complexes. It is natural to expect they are related. Indeed, firstly these complexes have the same cohomology, being homotopy equivalent (Proposition 2.2), and in particular the determinants  $\det H^*(\Omega^*M, d)$  and  $\det H^*(\mathcal{E}, d_H)$  are canonically isomorphic. Moreover, from the point of view of spectral geometry, the non-exploding part of the Hodge-de Rham spectrum converges towards the contact complex spectrum when one takes the sub-Riemannian (or diabatic) limit  $\varepsilon \searrow 0$  of calibrated metrics  $g_\varepsilon = d\theta(\cdot, J\cdot) + \varepsilon^{-1}\theta^2$ ; see [38, 7]. Note that the classical Ray–Singer metric stays constant in this limit, being independent of the metric on  $M$ .

However, we cannot prove equality of metrics in general, but only on particular contact manifolds called *CR Seifert manifolds* in [7]. These are CR manifolds  $(M, H, J)$  of dimension 3 admitting a transverse locally free circle action preserving the CR structure  $(H, J)$ ; see Definition 4.1. The generator  $T = d/dt$  of the circle action is the Reeb field of an invariant contact form  $\theta$ . On such a manifold  $M$ , endowed with any unitary representation  $\rho : \pi_1(M) \rightarrow U(N)$ , Theorem 4.2 states that the two Ray–Singer metrics of the twisted de Rham and contact complexes coincide.

In the last part of this work we analyse in detail the torsion function  $\kappa(s)$  for CR Seifert manifolds. It first turns out that  $\kappa(s)$  is a dynamical function in this case, depending only on the topology of  $M$ , together with the holonomies of the representation along the various primitive closed orbits of the circle action, as stated in formula (46) and Theorem 5.4.

Specialising to  $s = 0$  leads in Theorem 5.7 to an explicit formula for the Ray–Singer torsion and metric, twisted by any unitary representation. This Lefschetz-type formula extends a formula given by David Fried [17] in the acyclic case, i.e.  $H^*(M, \rho) = \{0\}$ , via topological methods and Reidemeister–Franz torsion. Fried interprets it as the identity

$$(4) \quad T_{RS}(M, \rho) = |\exp(Z_F(0))|,$$

where  $Z_F(0)$  stands for the analytic continuation at  $s = 0$  of the dynamical function

$$Z_F(s) = - \sum_C \text{ind}(C) \text{Tr}(\rho(C)) e^{-s\ell(C)}.$$

Here the sum describes all free homotopical classes of closed orbits of the Reeb flow  $T$ ,  $\text{ind}(C)$  denotes its Fuller index (Proposition 5.8),  $\ell(C)$  its length and  $\rho(C)$  its holonomy.

Our approach leads to another viewpoint on (4). Namely, we show in Theorem 5.9 and (66) that our purely spectral torsion function  $\kappa(s)$  may be seen as a dynamical zeta function, *in its whole*. Indeed it holds that

$$\Gamma(s)(\kappa(s) - \kappa(M, \rho)) = \frac{2^{1-2s}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} - s\right) Z_\rho(2s),$$

for

$$Z_\rho(s) = \sum_C \text{ind}(C) {}^R\text{Tr}(\rho(C)) \ell(C)^s,$$

where again the sum runs over all free homotopical classes of closed orbits of the Reeb flow,  ${}^R\text{Tr}$  is the real part of the trace, and  $\kappa(M, \rho) = 2 \dim H^0(M, \rho) - \dim H^1(M, \rho)$  is a purely cohomological term.

This Selberg-type trace formula also has a counterpart at the level of heat kernels. Indeed let  $\text{Tr}_\kappa(e^{-t\Delta}) = 2 \text{Tr}(e^{t\Delta_0}) - \text{Tr}(e^{-t\Delta_1})$ ; then we show in Theorem 5.10 that

$$\text{Tr}_\kappa(e^{-t\Delta}) = \dim V \frac{\sqrt{\pi} \chi(\Sigma)}{\sqrt{t}} + \frac{1}{\sqrt{\pi t}} \sum_C \ell(C) \text{ind}(C) {}^R\text{Tr}(\rho(C)) e^{-\ell(C)^2/4t},$$

where  $\chi(\Sigma)$  is the rational Euler class of the quotient surface orbifold  $\Sigma = M/\mathbb{S}^1$ . Hence our torsion heat trace (of fourth-order Laplacians) coincides with a dynamical theta function. This has some surprising consequences for the small time development of  $\text{Tr}_\kappa(e^{-t\Delta})$  on CR Seifert manifolds, but also on general 3-dimensional contact manifolds, as given in Corollaries 5.12 and 5.13.

The paper is organised as follows. In §2 we first review one construction of the contact complex  $(\mathcal{E}^*, d_H)$  and recall the Ray–Singer argument, from the viewpoint of the Ray–Singer metric on the determinant of the cohomology. We then adapt this argument to the contact complex, which leads us to define the analytic torsion  $T_C$ , a torsion function  $\kappa$  and a Ray–Singer metric  $\| \cdot \|_C$  on  $\det H(\mathcal{E}^*, d_H)$ .

In §3 we start the analytic study of this torsion. After reviewing relevant properties of hypoelliptic zeta functions and heat kernels, we establish variational formulae for  $\kappa(0)$ , the torsion  $T_C$  and the contact Ray–Singer metric  $\| \cdot \|_C$ . We then show that  $\kappa(0) = 0$  in dimension 3, and introduce corrections of the metric  $\| \cdot \|_C$  that give CR and contact invariants.

In §4 we compare Ray–Singer analytic torsion to ours and show that the two Ray–Singer metrics coincide on CR Seifert manifolds.

The final §5 is devoted to the study of the dynamical aspects of the torsion function of the contact complex, still on CR Seifert manifolds.

**Acknowledgements.** The second author thanks his PhD supervisor Prof. Kengo Hirachi for his expert guidance, and Prof. Raphaël Ponge for kindly explaining some of his results and for helpful comments on an earlier draft of parts of this paper. He is grateful to Prof. Robin Graham for his hospitality during the second author’s visit to the University of Washington in the summer of 2006. He thanks the organisers of the 2006 IMA Summer Program ‘Symmetries and Overdetermined Systems of Partial Differential Equations’ and 2006 ‘Seoul–Tokyo Conference in Mathematics’, where parts of this project were worked on and presented.

We are also grateful to Mike Eastwood, Robin Graham and Patrick Gérard for useful discussions. Finally, we benefited from stimulating and enlightening conversations with Jean-Michel Bismut, which led us to feel some coherence or correspondence between our results and his recent works [8, 12, 9] on Fried’s conjecture and the hypoelliptic Laplacian.

## 2. CONTACT ANALYTIC TORSION VIA A DETERMINANT BUNDLE

Ray and Singer [33] defined analytic torsion of the de Rham complex as an infinite-dimensional analogue of the Reidemeister–Franz torsion of finite simplicial complexes. Our purpose in this section is to describe their argument and adapt it to a similar complex defined on contact manifolds, the construction of which we now review.

**2.1. Contact complex.** Let  $(M, H)$  be a smooth orientable contact manifold of dimension  $2n + 1$ . This means that the smooth *contact distribution*  $H \subset TM$  is given as the null space of a globally defined 1-form, called a *contact form*, satisfying the condition of maximal non-integrability  $\theta \wedge (d\theta)^n \neq 0$ . The contact forms comprise an equivalence class under multiplication by smooth non-vanishing functions.

The *contact complex* ([36, 37]) is a refinement of the de Rham complex on contact manifolds defined as follows. Let  $\Omega^*M$  denote sections of the graded bundle of smooth differential forms on  $M$ ,  $\mathcal{I}$  the ideal in  $\Omega^*M$  generated by  $\theta$  and  $d\theta$ , and  $\mathcal{J}$  the ideal in  $\Omega^*M$  consisting of elements annihilated by  $\theta$  and  $d\theta$ . One verifies that  $\mathcal{I}^k = \Omega^k M$  for  $k \geq n + 1$ ,  $\mathcal{J}^k = 0$  for  $k \leq n$ , and that the de Rham exterior derivative  $d$  naturally induces operators  $d_H$  to form two complexes

$$\Omega^0 M \xrightarrow{d_H} \Omega^1 M / \mathcal{I}^1 \xrightarrow{d_H} \dots \xrightarrow{d_H} \Omega^n M / \mathcal{I}^n$$

and

$$\mathcal{J}^{n+1} \xrightarrow{d_H} \mathcal{J}^{n+2} \xrightarrow{d_H} \dots \xrightarrow{d_H} \mathcal{J}^{2n+1}.$$

It is clear that these two complexes are defined independently of the choice of  $\theta$ . These two complexes are joined by a second-order differential operator  $D : \Omega^n M / \mathcal{I}^n \rightarrow \mathcal{J}^{n+1}$  defined by setting  $D[\alpha] = d\beta$ , where  $\beta \in \Omega^n M$  is defined by the following:

**Lemma 2.1** ([36, 37]). *Let  $\alpha \in \Omega^n M$ . Then there exists a unique  $\beta \in \Omega^n M$  such that  $\beta \equiv \alpha \pmod{\theta}$  and  $\theta \wedge d\beta = 0$ . Moreover  $d\beta \in \mathcal{J}^{n+1}$ , and if  $\alpha \in \mathcal{I}^n$  then  $d\beta = 0$ .*

One can show that  $D$  may in fact be defined independently of the choice of  $\theta$ . The contact complex is

$$\Omega^0 M \xrightarrow{d_H} \Omega^1 M / \mathcal{I}^1 \xrightarrow{d_H} \dots \xrightarrow{d_H} \Omega^n M / \mathcal{I}^n \xrightarrow{D} \mathcal{J}^{n+1} \xrightarrow{d_H} \mathcal{J}^{n+2} \xrightarrow{d_H} \dots \xrightarrow{d_H} \mathcal{J}^{2n+1}.$$

We will write  $\mathcal{E}^k$  to mean either  $\Omega^k M / \mathcal{I}^k$  or  $\mathcal{J}^k$ . We also have:

**Proposition 2.2** ([37, pg. 286]). *The contact complex forms a resolution of the constant sheaf  $\mathbb{R}$  and hence its cohomology coincides with the de Rham cohomology of  $M$ . Moreover the canonical projection  $\pi : \Omega^k M \rightarrow \Omega^k M / \mathcal{I}^k$  for  $k \leq n$  and injection  $i : \mathcal{J}^k \rightarrow \Omega^k M$  for  $k \geq n + 1$  induce an isomorphism between the two cohomologies.*

The arguments being purely local, these results also apply to twisted versions of the complex with a flat bundle, as coming from a representation  $\rho : \pi_1(M) \rightarrow U(N)$ .

It is a basic fact that the symplectic bundle  $(H, d\theta)$  admits a contractible homotopy class of *calibrated* almost complex structures, i.e.  $J \in \text{End}(H)$  is in this class if and only if  $J^2 = -1$  and the *Levi metric*  $d\theta(\cdot, J\cdot)$  is positive definite and Hermitian.

The *Reeb field* of  $\theta$  is the unique vector field  $T$  satisfying  $\theta(T) = 1$  and  $T \lrcorner d\theta = 0$ . Fixing a  $\theta$  and a  $J$ , we may define a Riemannian metric  $g$  on  $M$  by using the Levi metric on vectors in  $H$  and declaring that the Reeb field  $T$  is of unit-length and orthogonal to  $H$ , i.e.

$$g = d\theta(\cdot, J\cdot) + \theta^2.$$

With these choices, it is straightforward to identify the quotients of forms appearing in the lower-half of the contact complex with actual forms ([37, pg. 288]):

$$\Omega^k N / \mathcal{I}^k \cong \{\alpha \in \Omega^k N : T \lrcorner \alpha = 0 = \Lambda \alpha\},$$

where  $\Lambda$  is the adjoint of the operator  $L : \Omega^k N \rightarrow \Omega^{k+2} N$ ,  $L\alpha = d\theta \wedge \alpha$ .

We henceforth assume that  $M$  is compact. With the identification above we now have an  $L^2$  inner product defined on the contact complex. Let  $\delta_H$ ,  $D^*$  denote the formal adjoint operators. It is straightforward to verify that

$$(5) \quad \delta_H|_{\mathcal{E}^k} = (-1)^k * d_H^*, \quad D^* = (-1)^{n+1} * D^*,$$

where  $* : \mathcal{E}^k \xrightarrow{\cong} \mathcal{E}^{2n+1-k}$  is the usual Hodge  $*$  operator.

As a last comment here, we mention there exist other approaches to this elementary construction of the contact complex. One possibility is via spectral sequence considerations, using a canonical filtration by Heisenberg weight of forms  $\Omega^* M$ ; see [38, §3] and [7] where this approach is used in the study of the sub-Riemannian (adiabatic) limit of the Hodge–de Rham spectrum. Another interesting viewpoint is to consider the contact complex as a curved version of a Bernstein–Gelfand–Gelfand complex in parabolic geometry; see [2, §8.1] for such a presentation on the 3-dimensional Heisenberg group.

**2.2. Determinant bundles, metrics and Reidemeister–Franz torsion.** We follow the presentation of Bismut and Zhang [13] to define the Reidemeister–Franz torsion of a finite-dimensional complex.

Let  $E$  be a finite-dimensional real vector space, and define the line

$$\det E = \wedge^{\max} E.$$

A useful convention here is to set  $\det\{0\} = \mathbb{R}$  (compatible with  $\det(E \oplus F) = \det E \otimes \det F$ ). If  $\lambda$  is a line, let  $\lambda^{-1} = \lambda^*$  be its dual line. Then  $\lambda \otimes \lambda^{-1} = \text{End}(\lambda) = \mathbb{R} \text{Id}$  is canonically isomorphic to  $\mathbb{R}$ .

One extends these notions to a finite-dimensional complex. Let

$$(E, d) : 0 \longrightarrow E_0 \xrightarrow{d} E_1 \xrightarrow{d} \cdots \xrightarrow{d} E_n \longrightarrow 0$$

be such a complex and  $H^*(E, d)$  its cohomology. Define

$$\det E = \bigotimes_{k=0}^n (\det E_k)^{(-1)^k}$$

and

$$\det(H^*(E, d)) = \bigotimes_{k=0}^n (\det H^k(E, d))^{(-1)^k}.$$

**Proposition 2.3** (Knudsen–Mumford [25]). *The lines  $\det E$  and  $\det(H^*(E, d))$  are canonically isomorphic.*

*Proof.* We include the proof as the explicit form of the isomorphism will be useful below. We follow [10]. Suppose first that  $H^*(E, d) = \{0\}$  so that  $\det H^*(E, d) = \mathbb{R}$ . Then we need to find a canonical section of  $\det E$ . Let  $N_k = \dim E_k$ . Pick a non-vanishing element

$$(6) \quad s_0 = e_1 \wedge e_2 \wedge \cdots \wedge e_{N_0} \in \wedge^{N_0} E_0 = \det E_0;$$

then

$$(7) \quad ds_0 = de_1 \wedge de_2 \wedge \cdots \wedge de_{N_0} \in \wedge^{N_0} E_1$$

is non-vanishing since  $d : E_0 \rightarrow E_1$  is injective.

Next pick  $s_1 \in \wedge^{N_1 - N_0} E_1$  such that  $ds_0 \wedge s_1$  generates  $\det E_1$ , and so on, taking  $s_k \in \wedge^{N_k - \cdots + (-1)^k N_0} E_k$  such that  $ds_{k-1} \wedge s_k$  generates  $\det E_k$ . Consider now

$$(8) \quad S(E, d) = s_0 \otimes (ds_0 \wedge s_1)^{-1} \otimes (ds_1 \wedge s_2) \otimes \cdots \otimes (ds_{n-1})^{(-1)^n} \in \det E.$$

It is clear that the class  $S(E, d)$  is non-zero and does not depend on the choices of  $s_k$  for  $k = 0, \dots, n$ , completing the proof of the proposition in the acyclic case.

For the general case, observe that the determinants of the short exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & dE_k & \rightarrow & \ker d|_{E_{k+1}} & \rightarrow & H^{k+1}(E, d) \rightarrow 0 \\ 0 & \rightarrow & \ker d|_{E_{k+1}} & \rightarrow & E_{k+1} & \rightarrow & dE_{k+1} \rightarrow 0 \end{array}$$

each have a canonical element, as was just shown above. So we have canonical isomorphisms

$$\begin{aligned} \det(\ker d|_{E_{k+1}}) &\cong \det(dE_k) \otimes \det(H^{k+1}(E, d)) \\ \det(E_{k+1}) &\cong \det(\ker d|_{E_{k+1}}) \otimes \det(dE_{k+1}), \end{aligned}$$

and then

$$\det(E_{k+1}) \cong \det(dE_k) \otimes \det(H^{k+1}(E, d)) \otimes \det(dE_{k+1}).$$

Finally taking tensor products over  $k$  gives

$$\det E_0 \otimes (\det E_1)^{-1} \otimes \det E_2 \otimes \cdots \cong \det H^0 \otimes (\det H^1)^{-1} \otimes \det H^2 \otimes \cdots$$

canonically. □

Suppose now  $E$  is given a metric  $g$ . Hence  $\det E$  has an induced metric. One can then define a metric on  $\det H^*(E, d)$  by

$$\| \cdot \|_{\det H^*(E, d)} = \| \cdot \|_{\det E},$$

using the canonical isomorphism given by Proposition 2.3.

Let  $d^* = \delta$  be the adjoint of  $d$ . By finite-dimensional Hodge theory,  $H^*(E, d)$  identifies with the harmonic forms

$$\mathcal{H}^*(E, d) = \{s \in E \mid ds = d^*s = 0\}.$$

By their inclusion in  $E$ , the harmonic forms inherit a metric. We then have a second metric  $\| \cdot \|_{\det H^*(E, d)}$  on  $\det H^*(E, d)$  via the above identification.

**Definition 2.4.** The *torsion* of the complex  $(E, d)$  with metric  $g$  is the ratio

$$\tau(E, d, g) = \frac{\| \det H^*(E, d) \|}{\| \det H^*(E, d) \|}.$$

*Remark 2.5.* Note that this definition of torsion, given in [13, §2] for instance, is quite natural at the metric level, but actually leads to the *inverse* of the original Reidemeister–Franz torsion (or  $R$ -torsion), i.e.

$$\tau_R(E, d, g) = 1/\tau(E, d, g);$$

see [33, §1] and [17, §2].

One can be more explicit using the proof of Proposition 2.3. Consider

$$F = \mathcal{H}^*(E, d)^\perp.$$

The complex  $(F, d)$  is acyclic so we can construct the canonical class  $S(F, d)$  as in (8).

**Proposition 2.6.** *For any generators  $s_i \in \det(\ker d)^\perp$  we have*

$$(9) \quad \tau(E, d, g) = \|S(F, d)\|_{\det F} = \left( \frac{\|s_0\|}{\|ds_0\|} \right) \times \left( \frac{\|s_1\|}{\|ds_1\|} \right)^{-1} \times \cdots \times \left( \frac{\|s_{n-1}\|}{\|ds_{n-1}\|} \right)^{(-1)^{n-1}}.$$

*Proof.* The splitting  $E = \mathcal{H}^*(E, d) \oplus F$  induces the canonical isomorphism

$$\begin{aligned} \det \mathcal{H}^*(E, d) &\xrightarrow{\cong} \det E = \det \mathcal{H}^*(E, d) \otimes \det F \\ s &\longmapsto s \otimes S(F, d). \end{aligned}$$

Then

$$\|s \otimes S(F, d)\|_{\det E} = \|s\|_{\det \mathcal{H}^*(E, d)} \|S(F, d)\|_{\det F}$$

and by Definition 2.4

$$\begin{aligned} \tau(E, d, g) &= \|S(F, d)\|_{\det F} \\ &= \|s_0\| \times \|ds_0 \wedge s_1\|^{-1} \times \|ds_1 \wedge s_2\| \times \cdots \times \|ds_{n-1}\|^{(-1)^n} \\ &= \left( \frac{\|s_0\|}{\|ds_0\|} \right) \times \left( \frac{\|s_1\|}{\|ds_1\|} \right)^{-1} \times \cdots \times \left( \frac{\|s_{n-1}\|}{\|ds_{n-1}\|} \right)^{(-1)^{n-1}}, \end{aligned}$$

since if  $s_i \in \det(\ker d)^\perp$ ,  $\|ds_i \wedge s_{i+1}\| = \|ds_i\| \|s_{i+1}\|$ . □

At this point in the Riemannian case ([33, 13]) one can guess the correct formula for analytic torsion by considering the Reidemeister–Franz torsion of finite-dimensional subcomplexes that approximate the infinite-dimensional de Rham complex  $(\Omega^*, d)$ . A natural choice of subcomplexes is obtained here by taking cut-off de Rham complexes using the spectrum of the Hodge–de Rham Laplacian  $\Delta = d\delta + \delta d$ ; that is, one considers the energy levels  $\Omega_{[0, \lambda]}^* = \{\Delta \leq \lambda\}$ . One then expresses (9) using the determinants of  $\Delta$  on  $(\Omega_{[0, \lambda]}^*, d)$  and finally as combinations of differentiated zeta functions  $\zeta'(\Delta)(0)$  in the limiting infinite-dimensional case. We carry out this procedure for the contact complex next.



**2.3. Defining a contact analytic torsion.** Consider now the contact complex

$$(\mathcal{E}, d_H) : \mathcal{E}^0 \xrightarrow{d_H} \mathcal{E}^1 \xrightarrow{d_H} \dots \xrightarrow{d_H} \mathcal{E}^n \xrightarrow{D} \mathcal{E}^{n+1} \xrightarrow{d_H} \dots \xrightarrow{d_H} \mathcal{E}^{2n+1}.$$

We want to define finite-dimensional subcomplexes of the contact complex via finite energy cut-offs for a certain Laplacian  $\Delta$ .

We shall use the following uniformly fourth-order Laplacian:

$$(10) \quad \Delta := \begin{cases} (d_H \delta_H + \delta_H d_H)^2 & \text{on } \mathcal{E}^k \text{ for } k \neq n, n+1 \\ (d_H \delta_H)^2 + D^* D & \text{on } \mathcal{E}^n \\ DD^* + (\delta_H d_H)^2 & \text{on } \mathcal{E}^{n+1}. \end{cases}$$

We denote by  $\Delta_k$  the restriction of  $\Delta$  to  $\mathcal{E}^k$ . The rationale behind our choice of  $\Delta$  is as follows. In middle degrees, because  $D$  is second-order, one needs to square the terms involving  $d_H$  so that  $\Delta$  has certain good analytical properties. In particular,  $\Delta$  is maximally hypoelliptic and invertible in the Heisenberg symbolic calculus, while the standard combinations  $d_H \delta_H + D^* D$  and  $DD^* + d_H \delta_H$  are not; see §3.1 below. If we then consider the spectral spaces  $E_{[0, \lambda]}^n = \{\Delta_n \leq \lambda\}$  as successive finite-dimensional approximations of  $\mathcal{E}^n$ , in order to include  $E_{[0, \lambda]}^n$  in a finite-dimensional subcomplex of  $(\mathcal{E}^*, d)$  we need the Laplacians outside middle degree to be fourth-order also.

*Remark 2.7.* Note that  $\Delta$  is different to the Laplacian  $\Delta_Q$  defined in [37]. The latter was defined with nice algebraic properties, namely commutativity with  $J$  when  $\mathcal{L}_T J = 0$ . On the other hand, observe that  $\Delta$  commutes with  $d_H, D$  and their adjoints, which, as we shall see in §3.2, is essential for analytic torsion of the contact complex having the correct variational behaviour. Moreover note that  $\Delta$  is the Laplacian appearing in the sub-Riemannian limit ([38]).

Next let us set

$$E_{[0, \lambda]}^* = \bigoplus_{k=0}^{2n+1} \{\Delta_k \leq \lambda\}.$$

These are finite-dimensional subcomplexes of the contact complex, each of which splits into a direct sum of eigenspaces  $E_\mu^*$  of  $\Delta$  for  $0 \leq \mu \leq \lambda$ .

**Proposition 2.8.** *The torsion of  $(E_{[0, \lambda]}^*, d_H)$  is*

$$(11) \quad \tau(E_{[0, \lambda]}^*, d_H) = \prod_{k=0}^{2n+1} \det(\Delta_k | E_{[0, \lambda]}^*)^{(-1)^k w(k)/4},$$

where

$$(12) \quad w(k) = \begin{cases} k & \text{if } k \leq n \\ k+1 & \text{if } k > n. \end{cases}$$

*Remark 2.9.* This  $w(k)$  is the natural weight of  $\mathcal{E}^k$  in the contact complex; see §3.1 and [38, §3].

*Proof.* We use Proposition 2.6. By the orthogonal splitting  $E_{[0,\lambda]}^* = \oplus_{\mu \leq \lambda} E_\mu^*$  and (9) we have

$$(13) \quad \tau(E_{[0,\lambda]}^*, d_H) = \prod_{0 < \mu \leq \lambda} \|S(E_\mu, d_H)\|.$$

Given  $\mu \in ]0, \lambda]$  we pick an orthogonal basis  $(v_1, \dots, v_{N_k})$  of each eigenspace

$$E_\mu^k \cap (\ker d)^\perp = F_\mu^k$$

and choose the elements  $s_i = v_1 \wedge \dots \wedge v_{N_k} \in \det F_\mu^k$  as in (6). Here  $N_k = \dim F_\mu^k$ . Now by (7) and orthogonality

$$\begin{aligned} \|ds_i\| &= \|dv_1 \wedge dv_2 \wedge \dots \wedge dv_{N_k}\| \\ &= \|dv_1\| \times \|dv_2\| \times \dots \times \|dv_{N_k}\| \\ &= \begin{cases} \mu^{N_k/4} \|v_1\| \times \dots \times \|v_{N_k}\| = \mu^{N_k/4} \|s_i\| & \text{for } k \neq n \\ \mu^{N_k/2} \|v_1\| \times \dots \times \|v_{N_k}\| = \mu^{N_k/2} \|s_i\| & \text{for } k = n. \end{cases} \end{aligned}$$

Therefore from (9)

$$(14) \quad \|S(E_\mu, d_H)\| = \mu^{M_\mu}$$

with

$$4M_\mu = -N_0 + N_1 - N_2 \dots + (-1)^{n+1} 2N_n + (-1)^n N_{n+1} \dots - N_{2n}.$$

Let  $e_k = \dim E_\mu^k$ . From  $E_\mu^k = F_\mu^k \oplus dF_\mu^{k-1}$ , one has  $e_k = N_k + N_{k-1}$  and therefore

$$N_k = e_k - e_{k-1} + \dots + (-1)^k e_0.$$

This leads to

$$\begin{aligned} 4M_\mu &= -(2n+2)e_0 + (2n+1)e_1 + \dots + (-1)^n (n+3)e_{n-1} \\ &\quad + (-1)^{n+1} (n+2)e_n + (-1)^{n+2} n e_{n+1} \\ &\quad + (-1)^{n+3} (n-1)e_{n+2} + \dots + e_{2n+1} \\ &= \sum_{k=0}^{2n+1} (-1)^k w(k) e_k, \end{aligned}$$

using that  $\sum_{k=0}^{2n+1} (-1)^k e_k = 0$  for the acyclic complex  $(E_\mu^*, d_H)$ .

As  $\Delta = \mu$  on  $E_\mu^*$ , equation (14) reads

$$\|S(E_\mu, d_H)\|^4 = \prod_{k=0}^{2n+1} \det(\Delta_k | E_\mu)^{(-1)^k w(k)},$$

leading by (13) to the assertion of the proposition.  $\square$

We finally introduce zeta functions of the contact Laplacian. If  $\text{spec}^*(\Delta_k)$  denotes the non-zero spectrum of  $\Delta_k$  on  $\mathcal{E}_k$ , then we take

$$\zeta(\Delta_k)(s) = \dim H^k(\mathcal{E}, d_H) + \sum_{\lambda \in \text{spec}^*(\Delta_k)} \lambda^{-s}.$$

Note that by hypoellipticity (or Proposition 2.2)  $\dim H^k(\mathcal{E}, d_H)$  is finite. By the results in §3.1 below,  $\zeta(\Delta_k)(s)$  admits a meromorphic extension to  $\mathbb{C}$  that is regular at  $s = 0$ . On each subcomplex  $(E_{[0, \lambda]}^*, d_H)$  we then have

$$\begin{aligned} \zeta'(\Delta_k | E_{[0, \lambda]})(0) &= - \sum_{\mu \in \text{spec}^*(\Delta_k) \cap ]0, \lambda]} \ln \mu \\ &= - \ln \det(\Delta_k | E_{[0, \lambda]}). \end{aligned}$$

Thus formula (11) for the torsion of  $(E_{[0, \lambda]}^*, d_H)$  can be written for  $\lambda > 0$  as

$$(15) \quad \ln \tau(E_{[0, \lambda]}^*, d_H) = \frac{1}{4} \sum_{k=0}^{2n+1} (-1)^{k+1} w(k) \zeta'(\Delta_k | E_{[0, \lambda]}^*)(0).$$

We thus speculate in extending this formula to the whole contact complex by defining the *analytic torsion of the contact complex* as

$$(16) \quad \ln T_C = \frac{1}{4} \sum_{k=0}^{2n+1} (-1)^{k+1} w(k) \zeta'(\Delta_k)(0).$$

This formula is very similar to that of Ray–Singer analytic torsion  $T_{RS}$  in the Riemannian setting. Namely, from [33, Definition 1.6], in dimension  $N$

$$(17) \quad \ln T_{RS} = \frac{1}{2} \sum_{k=0}^N (-1)^k k \zeta'(\Delta_k)(0),$$

for Hodge–de Rham Laplacians  $\Delta_k$ . Note however the sign convention:  $T_{RS}$  coincides with Reidemeister–Franz torsion  $\tau_R$  on finite-dimensional cut-off de Rham complexes, while our  $T_C$  leads to the inverse; see Remark 2.5.

By analogy with Definition 2.4 and [32, 13], we also define a *contact complex Ray–Singer metric* on  $\det H^*(\mathcal{E}, d_H)$  by setting

$$(18) \quad \| \quad \|_C = T_C \quad | \quad |_{L^2(\mathcal{E})}.$$

Here  $| \quad |_{L^2(\mathcal{E})}$  is the  $L^2$  metric induced on  $\det H^*(\mathcal{E}, d_H)$  by identification of  $H^*(\mathcal{E}, d_H)$  with harmonic forms  $\mathcal{H}^*(\mathcal{E}, d_H) \subset \mathcal{E}^*$ .

Again, note that the Ray–Singer metric on the de Rham determinant  $\det H^*(\Omega^*M, d)$  reads instead

$$(19) \quad \| \quad \|_{RS} = (T_{RS})^{-1} | \quad |_{L^2(\Omega^*M)}.$$

More generally, we can twist the contact complex with a flat bundle and then define the analytic torsion of this twisted contact complex  $(\mathcal{E}_\rho^*, d_H)$ . Indeed let  $\rho : \pi_1(M) \rightarrow U(N)$  be a unitary representation on  $\mathbb{C}^N$ . Associated to  $\rho$  is an Hermitian complex rank  $N$  vector bundle  $V_\rho$  equipped with a canonical metric-preserving flat connection  $\nabla_\rho$ . One sets  $\mathcal{E}_\rho = \mathcal{E} \otimes V_\rho$  and  $d_H(\alpha \otimes s) = d_H \alpha \otimes s$  for parallel  $s$ . From this we may define the contact analytic torsion  $T_C(\rho)$  with associated contact complex Ray–Singer metric on  $\det H^*(\mathcal{E}_\rho, d_H)$ .

The conciseness of notations  $T_C$  and  $T_C(\rho)$  should not be misleading. The (twisted) contact complex only depends on the contact structure  $H$  on  $M$  (and  $\rho$ ), but the spectral invariants

$T_C$  and  $T_C(\rho)$  also depend on the choices of a contact form  $\theta$  and complex structure  $J$ , both being used in the metric  $g$ .

Since we have defined this analytic torsion through algebraic and formal considerations around Reidemeister–Franz torsion, we now need to study its analytical properties. That is the purpose of the next section.

### 3. HEAT KERNELS AND VARIATIONAL BEHAVIOUR OF THE TORSION

We first gather some properties of zeta functions and the heat development of hypoelliptic operators such as the Laplacian of the contact complex.

**3.1. Heat kernels and zeta functions for hypoelliptic operators.** The Laplacian  $\Delta$  for the contact complex is not elliptic. However there is a (substantially more intricate) symbolic calculus that can be applied to it to obtain results on heat kernels qualitatively analogous to the elliptic case. This calculus is called the *Heisenberg calculus* and was introduced by Beals–Greiner [3] and Taylor [44]. A short account of its properties may be found in [21], and its use for the contact complex has been presented by Julg and Kasparov in [23, §5]. This calculus has also been developed in a more general setting that includes the contact case; see the presentation of Ponge [30] for details and further references to the literature. Here we just briefly sketch the results that we shall need in the sequel.

**Proposition 3.1** (Ponge [30, Ch. 5]). *Let  $\mathcal{V}$  be a vector bundle over a compact contact manifold  $(M, H)$  of dimension  $2n + 1$ . Let  $P : C^\infty(M, \mathcal{V}) \rightarrow C^\infty(M, \mathcal{V})$  be a differential operator of even Heisenberg order  $v$  that is selfadjoint and bounded from below. If  $P$  satisfies the Rockland condition at every point then the principal symbol of  $P + \partial_t$  is an invertible Volterra–Heisenberg symbol and as  $t \searrow 0$  the heat kernel  $k_t(x, x)$  of  $P$  on the diagonal has the following asymptotics in  $C^\infty(M, (\text{End } \mathcal{V}) \otimes |\Lambda|(M))$ :*

$$k_t(x, x) \sim \sum_{j=0}^{\infty} t^{\frac{2(j-n-1)}{v}} a_j(P)(x).$$

Some explanation about the proposition is in order. The *Heisenberg order* of  $P$  is defined by assuming that a derivative in the direction of the Reeb field  $T$  has weight 2, while derivatives in the direction of the contact distribution  $H$  have weight 1. The *Rockland condition* is a representation-theoretic condition defined in [30, Definition 3.3.8]. (The original formulation is due to Rockland [34].) An operator that satisfies this condition is hypoelliptic, in the sense of [30, Proposition 3.3.2]. Invertibility of an operator in the *Volterra–Heisenberg calculus* is explained in [30, Ch. 5].

The next result describes the properties of the zeta function in this contact setting. For the non-negative operators  $P$  we are concerned with, the result follows from Proposition 3.1 by a classical argument using the Mellin transform of the heat kernel.

**Proposition 3.2** (Ponge [31, §4]). *Let  $P$  be as in Proposition 3.1. Then the zeta function*

$$\zeta(P)(s) = \dim \ker P + \text{Tr}^*(P^{-s}), s \in \mathbb{C},$$

is a well-defined holomorphic function for  $\operatorname{Re}(s) \gg 1$  and admits a meromorphic extension to  $\mathbb{C}$  with at worst simple poles occurring at  $s \in \mathcal{S} = \{\frac{2(n+1-j)}{v} \mid j \in \mathbb{N}\} \setminus (-\mathbb{N})$ . Moreover

$$\zeta(P)(0) = \int_M \operatorname{tr}(a_{n+1}(P)) \theta \wedge (d\theta)^n$$

is the constant term in the development of  $\operatorname{Tr}(e^{-tP})$  as  $t \searrow 0$ .

Now by [37, p. 300] and the definition ([30, Definition 3.3.8]) of the Rockland condition, or else by Julg and Kasparov's work [23, §5], the fourth-order Laplacian  $\Delta$  on the contact complex (twisted with a flat bundle) satisfies the Rockland condition, hence Propositions 3.1 and 3.2 apply to it.

*Remark 3.3.* Our convention is to include the eigenvalue 0 in the definition of the zeta function of a non-negative hypoelliptic operator  $P$ . This means  $\zeta(P)(0)$  consists of (the integral of) purely local terms for  $P$ .

**3.2. Variational behaviour of the analytic torsion.** We consider the variation of contact analytic torsion and Ray–Singer metric in the direction of an arbitrary line of pairs  $(\theta_\varepsilon, J_\varepsilon)$  of contact form and calibrated almost complex structure for  $(M, H)$ . We shall see that the variation of the Ray–Singer metric is given entirely by local terms. Indeed this may be viewed as a necessary and sufficient condition for correctly defining an analytic torsion; see the approach of Branson [14].

First we define from (16) the *contact torsion function* by

$$(20) \quad \kappa(s) = \frac{1}{2} \sum_{k=0}^{2n+1} (-1)^{k+1} w(k) \zeta(\Delta_k)(s),$$

with  $w(k)$  as in (12). Then the analytic torsion of the contact complex reads

$$T_C = \exp\left(\frac{1}{2} \kappa'(0)\right).$$

For simplicity, in this section we suppress the  $C$  from the notation, as well as the representation  $\rho$ , although all results stand for the twisted torsions and metrics as well.

**Theorem 3.4.** *Let a  $\bullet$  superscript denote first variation  $(d/d\varepsilon)|_{\varepsilon=0}$ .*

- (1) *One has  $\kappa(0)^\bullet = 0$ , so that  $\kappa(0)$  is a contact invariant.*
- (2) *The variation of the analytic torsion  $T$  is given by*

$$(\ln T)^\bullet = \sum_{k=0}^n (-1)^k \left( \int_M \operatorname{tr}(\alpha a_{t^0, k}) \, \operatorname{dvol} - \operatorname{Tr}(\alpha \mathcal{P}_k) \right),$$

where  $\alpha = *^{-1} \bullet$ ,  $a_{t^0, k}$  is the  $t^0$  coefficient in the diagonal small-time asymptotic expansion of the heat kernel of  $\Delta_k$ ,  $\operatorname{dvol}$  is the volume form  $\theta \wedge (d\theta)^n$ , and  $\mathcal{P}_k$  is orthogonal projection onto the null-space of  $\Delta_k$ .

*Proof.* By Hodge  $*$  duality (for convenience we suppress the  $\varepsilon$  dependence)

$$\kappa(s) = \sum_{k=0}^n c_k \zeta(\Delta_k)(s)$$

with

$$c_k = (-1)^k (n+1-k).$$

By a Mellin transform set

$$f(s) = \Gamma(s)\kappa(s) = \sum_{k=0}^n c_k \int_0^{+\infty} t^{s-1} \operatorname{Tr}(e^{-t\Delta_k} - \mathcal{P}_k) dt + \Gamma(s) \sum_{k=0}^n c_k \dim H^k(\mathcal{E}, d_H),$$

where  $\mathcal{P}_k$  denotes orthogonal projection onto the null-space of  $\Delta_k$ .

**Lemma 3.5.** *It holds that*

$$(\operatorname{Tr}(e^{-t\Delta}))^\bullet = -t \operatorname{Tr}(\Delta^\bullet e^{-t\Delta}).$$

*Proof.* Duhamel's formula (see e.g. [35, Proposition 3.15]) implies that

$$\operatorname{Tr}(e^{-t(\Delta+\varepsilon\Delta^\bullet+O(\varepsilon^2))}) - \operatorname{Tr}(e^{-t\Delta}) = - \int_0^t \operatorname{Tr}(e^{-(t-s)(\Delta+\varepsilon\Delta^\bullet+O(\varepsilon^2))} \varepsilon(\Delta^\bullet + O(\varepsilon)) e^{-s\Delta}) ds.$$

Dividing both sides by  $\varepsilon$  then letting  $\varepsilon \rightarrow 0$  gives

$$(\operatorname{Tr}(e^{-t\Delta}))^\bullet = - \int_0^t \operatorname{Tr}(e^{-(t-s)\Delta} \Delta^\bullet e^{-s\Delta}) ds.$$

Recalling that  $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ , for smoothing operators  $A, B$ , we have

$$(\operatorname{Tr}(e^{-t\Delta}))^\bullet = - \int_0^t \operatorname{Tr}(\Delta^\bullet e^{-s\Delta} e^{-(t-s)\Delta}) ds,$$

and the lemma follows.  $\square$

By Lemma 3.5

$$(21) \quad f(s)^\bullet = - \sum_{k=0}^n c_k \int_0^{+\infty} t^s \operatorname{Tr}(\Delta_k^\bullet e^{-t\Delta_k}),$$

since by hypoellipticity  $\operatorname{Tr}(\mathcal{P}_k) = \dim \ker \Delta_k = \dim H^k(\mathcal{E}, d_H)$  is certainly independent of  $\theta$  and  $J$ .

Setting  $\alpha = *^{-1}\bullet$ , one computes using (5) the variation of the Laplacian as

$$\Delta^\bullet = \begin{cases} -d_H \alpha \delta_H d_H \delta_H + d_H \delta_H \alpha d_H \delta_H - d_H \delta_H d_H \alpha \delta_H + d_H \delta_H d_H \delta_H \alpha \\ \quad - \alpha \delta_H d_H \delta_H d_H + \delta_H \alpha d_H \delta_H d_H - \delta_H d_H \alpha \delta_H d_H + \delta_H d_H \delta_H \alpha d_H \\ \quad \text{on } \mathcal{E}^k \text{ for } k = 0, \dots, n-1, \\ -d_H \alpha \delta_H d_H \delta_H + d_H \delta_H \alpha d_H \delta_H - d_H \delta_H d_H \alpha \delta_H + d_H \delta_H d_H \delta_H \alpha \\ \quad - \alpha D^* D + D^* \alpha D \\ \quad \text{on } \mathcal{E}^n. \end{cases}$$

A computation then shows that

$$(22) \quad \sum_{k=0}^n c_k \operatorname{Tr}(\Delta_k^\bullet e^{-t\Delta_k}) = 2 \sum_{k=0}^{n-1} \operatorname{Tr}((\alpha(c_k + c_{k-1})(d\delta)^2 - \alpha(c_k + c_{k+1})(\delta d)^2) e^{-t\Delta_k}) \\ + 2 \operatorname{Tr}((\alpha(c_n + c_{n-1})(d\delta)^2 - \alpha c_n D^* D) e^{-t\Delta_n}).$$

To move the  $\alpha$ 's to the front we have used the following facts: the heat kernel is a semigroup, implying  $e^{-t\Delta} = e^{-(t/2)\Delta}e^{-(t/2)\Delta}$ ; if operators  $A, B$  are smoothing then  $\text{Tr}(AB) = \text{Tr}(BA)$ ; and the Laplacian  $\Delta$  commutes with  $d_H, D$  and their adjoints. We have also used that

$$\text{Tr}(\alpha DD^* e^{-t\Delta_{n+1}}) = -\text{Tr}(\alpha D^* D e^{-t\Delta_n}),$$

as  $\alpha DD^* e^{-t\Delta_{n+1}} = -*^{-1}(\alpha D^* D e^{-t\Delta_n})*$ , which follows from (5) and  $(*)^\bullet = *\alpha + \alpha* = 0$ .

Simplifying (22) yields

$$\begin{aligned} \sum_{k=0}^n c_k \text{Tr}(\Delta_k^\bullet e^{-t\Delta_k}) &= 2 \sum_{k=0}^n (-1)^{k+1} \text{Tr}(\alpha \Delta_k e^{-t\Delta_k}) \\ &= -2 \frac{d}{dt} \sum_{k=0}^n (-1)^{k+1} \text{Tr}(\alpha e^{-t\Delta_k}). \end{aligned}$$

Hence after integrating by parts in (21),

$$(23) \quad f(s)^\bullet = 2s \sum_{k=0}^n (-1)^k \int_0^{+\infty} t^{s-1} \text{Tr}(\alpha e^{-t\Delta_k} - \alpha \mathcal{P}_k) dt.$$

We have included the  $\alpha \mathcal{P}_k$  term in the integrand to ensure that  $f(s)^\bullet$  is a well-defined meromorphic function on all  $\mathbb{C}$ .

As  $1/\Gamma(s) = O(s)$ ,  $f(s)^\bullet$  is regular at the origin. Now near  $s = 0$

$$f(s) = \Gamma(s)(\kappa(0) + s\kappa'(0) + O(s^2))$$

so

$$f(s)^\bullet = \Gamma(s)\kappa(0)^\bullet + s\Gamma(s)(\kappa'(0))^\bullet + O(s).$$

Since  $f(s)^\bullet$  is regular at the origin, letting  $s \rightarrow 0$  in the above equation gives

$$\kappa(0)^\bullet = 0 \quad \text{and} \quad (\kappa'(0))^\bullet = f(0)^\bullet.$$

This gives assertion (1) of Theorem 3.4. Assertion (2) follows from Proposition 3.2, after taking the inverse Mellin transform of (23).  $\square$

*Remark 3.6.* The previous proof actually shows that the torsion function

$$\kappa(s) = \sum_{k=0}^n (-1)^k (n+1-k) \zeta(\Delta_k)(s)$$

studied here is, up to a multiplicative factor, the *unique* combination of such zeta functions that leads to a variational formula like (23), i.e. local up to cohomological terms.

The variational formula for analytic torsion we obtained is more neatly expressed at the level of the Ray–Singer metric, since then the global term disappears. The next result is analogous to the variational formula for the Ray–Singer metric on Riemannian manifolds (see [13, Theorem 4.14] and [11, Theorem 1.18]).

**Corollary 3.7.** *Let  $\| \cdot \|_C$  denote the contact Ray–Singer metric on  $\det H^*(\mathcal{E}, d_H)$ .*

(1) *The following identity holds:*

$$(24) \quad (\ln \| \quad \|_C)^\bullet = \sum_{k=0}^n (-1)^k \int_M \operatorname{tr}(\alpha a_{t^0; k}) \theta \wedge (d\theta)^n.$$

(2) *Under conformal variations of the contact form  $(\theta_\varepsilon = e^{2\varepsilon\Upsilon}\theta, J_\varepsilon = J)$ , for a function  $\Upsilon$ , we have*

$$(\ln \| \quad \|_C)^\bullet = 2 \sum_{k=0}^n (-1)^k (n+1-k) \int_M \Upsilon \operatorname{tr}(a_{t^0; k}) \theta \wedge (d\theta)^n.$$

*Proof.* Recalling from (18) and §2 the definition of the Ray–Singer metric, we have that

$$(25) \quad \ln \| \quad \|_C^2 = 2 \ln T_C + \ln \| \quad \|_{L^2(\mathcal{E})}^2,$$

where the  $L^2$  metric  $\| \quad \|_{L^2(\mathcal{E})}$  is induced on  $\det H^*(\mathcal{E}, d_H)$  from the inner product on  $H^*(\mathcal{E}, d_H)$  defined by

$$\langle [u], [v] \rangle_{\theta, J} = \int_M \mathcal{P}u \wedge * \mathcal{P}v.$$

But for the orthogonal projection  $\mathcal{P}$  onto harmonic forms  $\mathcal{H}^*(\mathcal{E}, d_H)$ , one checks that  $\mathcal{P}^\bullet$  takes  $\mathcal{H}^*(\mathcal{E}, d_H)$  to its orthogonal complement. Thus

$$(26) \quad \langle [u], [v] \rangle^\bullet = \int_M \mathcal{P}u \wedge (*^\bullet) \mathcal{P}v = \langle [u], \alpha[v] \rangle.$$

If we use Hodge  $*$  duality in the definition of  $\det H^*(\mathcal{E}, d_H)$ , then take an orthonormal basis of each  $\mathcal{H}^k(\mathcal{E}, d_H)$ ,  $k = 0, \dots, n$ , and finally use (26), it is easy to see that

$$(\ln \| \quad \|_C)^\bullet = \sum_{k=0}^n (-1)^k \operatorname{Tr}(\alpha \mathcal{P}_k).$$

This together with (25) and Theorem 3.4 completes the proof of (1).

Assertion (2) follows immediately from (1), since for conformal variations it is straightforward to check that on  $\mathcal{E}^k$

$$\alpha = *^{-1} *^\bullet = 2(n+1-k) \Upsilon \operatorname{Id}.$$

□

Note that setting  $\Upsilon \equiv 1$  in Corollary 3.7 (2), i.e. performing a constant rescaling  $\theta \mapsto e^{2\varepsilon}\theta$ , yields

$$(27) \quad (\| \quad \|_C)^\bullet = 2\kappa(0).$$

In particular, if the contact invariant  $\kappa(0) \neq 0$ , then we could not hope for any invariance of Ray–Singer metric. Note that by definition

$$\begin{aligned} \kappa(0) &= \sum_{k=0}^n (-1)^k (n+1-k) \zeta(\Delta_k)(0) \\ &= \sum_{k=0}^n (-1)^k (n+1-k) \int_M \operatorname{tr}(a_{t^0; k}) \theta \wedge (d\theta)^n, \end{aligned}$$



by Proposition 3.2, where again  $a_{t^0;k}$  is the constant  $t^0$  coefficient in the diagonal small-time asymptotic expansion of the heat kernel of  $\Delta_k$ . Therefore  $\kappa(0)$  is an integral over  $M$  of local curvature data, namely

$$(28) \quad \kappa(0) = \int_M P_n(R, A, T, \nabla) \, \text{dvol},$$

for some universal polynomial in Tanaka–Tanno–Webster ([42, 43, 45]) curvature  $R$ , torsion  $A$ , Tanno’s tensor  $T = \nabla J$ , and their covariant derivatives.

We show in Corollary 3.8 below that, in dimension 3 ( $n = 1$ ),  $\kappa(0)$  vanishes identically. Whether contact invariants such as  $\kappa(0)$  vanish in all dimensions is an open problem. For further discussion in the contact case see [40, §7], and [7, Remark 9.3] for a similar problem arising for the eta function of the contact complex.

**3.3. CR/contact invariants in dimension 3.** In dimension 3, besides the vanishing of  $\kappa(0)$  we mentioned, we can also obtain more explicit variational formulae, and get CR/contact-invariant corrections to the contact Ray–Singer metric.

**Corollary 3.8.** *On 3-dimensional contact manifolds:*

- (1) *It holds that  $\kappa(0) = 0$ , and thus  $\| \cdot \|_C$  is independent of a constant rescaling  $\theta \mapsto K\theta$ .*
- (2) *There exist universal constants  $C_1, C_2$  such that under a conformal variation ( $\theta^\varepsilon = e^{2\varepsilon\Upsilon}\theta, J_\varepsilon = J$ ) we have*

$$(\ln \| \cdot \|_C)^\bullet = \int_M \Upsilon(C_1 \Delta_H R + C_2 \text{Im } A_{11}^{11}) \theta \wedge d\theta,$$

where  $R, A, \Delta_H$  and a comma subscript denote respectively the scalar curvature, torsion, sub-Laplacian and covariant differentiation with respect to the Tanaka–Webster connection ([42, 45]) of  $(\theta, J)$ .

- (3) *Let  $C'_1 = -C_1/8$  and  $C'_2 = C_2/4$ , with  $C_1, C_2$  as above. Then*

$$\| \cdot \|_{\text{CR}} = \exp \left( C'_1 \int_M R^2 \theta \wedge d\theta + C'_2 \int_M |A|^2 \theta \wedge d\theta \right) \| \cdot \|_C$$

is a CR-invariant (i.e. independent of contact form) metric on  $\det H^*(\mathcal{E}, d_H)$ .

- (4) *There exist universal constants  $C_3, C_4$  such that both*

$$\| \cdot \|_H^\nu = \exp(C_3 \nu(M)) \| \cdot \|_{\text{CR}} \quad \text{and} \quad \| \cdot \|_H^D = \exp(C_4 \bar{\eta}(D^*)) \| \cdot \|_{\text{CR}}$$

are contact-invariant metrics, with  $\nu(M)$  the  $\nu$ -invariant of Biquard–Herzlich [6], and  $\bar{\eta}(D^*)$  the CR-invariant correction to the pseudohermitian eta invariant  $\eta(D^*)$ ; see [7, Theorem 9.4].

*Proof.* We complexify  $H$  and work in a local frame  $\{Z_1, Z_{\bar{1}}\}$  and coframe  $\{\theta^1, \theta^{\bar{1}}\}$ , where  $\theta^1(T) = 0 = \theta^{\bar{1}}(T)$  ( $T$  the Reeb field). Under a constant scaling of contact form  $\hat{\theta} = K\theta$ , the relevant heat coefficient  $a_{t^0;k}$ , for  $k = 0, 1$ , scales as  $\text{tr}(\hat{a}_{t^0;k}) = K^{-2} \text{tr}(a_{t^0;k})$ . This is easily verified by an argument similar to that for [4, (6.48)]. Basic invariant theory (see e.g. [41, 7]) then tells us that  $\text{tr}(a_{t^0;k})$  must be a universal linear combination of

$$(29) \quad R^2, |A|^2, \Delta_H R, R_{,0} = 2 \text{Re } A_{11}^{11} \text{ and } \text{Im } A_{11}^{11}.$$

Now  $\kappa(0)$  is the integral of a linear combination of these terms, which is moreover independent of the choice of  $\theta$ . A familiar argument (see e.g. [7]) shows that the integral of a linear combination of  $R^2$  and  $|A|^2$  can never be contact-invariant. Thus  $\kappa(0)$  is the integral of a divergence, and hence vanishes. This, together with (27), proves assertion (1).

Consider now the differential of  $\ln \|\cdot\|_C$  under a conformal change of  $\theta$ . This may be seen as a real 1-form  $\alpha$  on the space  $\Theta$  of contact forms. By Corollary 3.7 (2) and (29) it can be written

$$\alpha_\theta(\Upsilon) = \int_M \Upsilon (c_1 R^2 + c_2 |A|^2 + c_3 \Delta_H R + c_4 R_{,0} + c_5 \operatorname{Im} A_{11},^{11}) \theta \wedge d\theta,$$

for some universal constants  $c_i$ . Here we identified the tangent space  $T_\theta \Theta$  with functions  $\Upsilon$  on  $M$ . By [7, Lemma 9.5], the general vanishing of such a 1-form on constant  $\Upsilon$  implies that  $c_1 = c_2 = 0$ , while the fact that  $\alpha$  is a *closed* form gives  $c_4 = 0$ ; see [7] for details. This proves assertion (2).

Also by (83)–(84) in [7, §9], one has

$$\frac{d}{d\Upsilon} \int_M R^2 \theta \wedge d\theta = 8 \int_M \Upsilon (\Delta_H R) \theta \wedge d\theta,$$

and

$$\frac{d}{d\Upsilon} \int_M |A|^2 \theta \wedge d\theta = -4 \int_M \Upsilon (\operatorname{Im} A_{11},^{11}) \theta \wedge d\theta,$$

leading to assertion (3).

Assertion (4) is proved similarly as for the case of the contact eta invariant in [7, §9]. The CR deformations (i.e. of  $J$ ) of the CR-invariants  $\nu$ ,  $\bar{\eta}(D^*)$  and  $\ln \|\cdot\|_{\text{CR}}$  are all given by multiples of

$$\int_M \langle Q, \dot{J} \rangle \theta \wedge d\theta,$$

where  $Q$  is Cartan's tensor; see [7, §9] for details.  $\square$

*Remark 3.9.* As may be seen from (24), in order to determinate the various universal constants in Corollary 3.8 and investigate whether  $\|\cdot\|_C$  has any contact-invariant properties, one needs to calculate the local coefficients of  $t^0$  in the diagonal small-time asymptotic expansion of the heat kernels of the fourth-order Laplacians we consider here. Formulae for calculating these coefficients are built into the pseudodifferential construction of the heat kernel, however implementing these in practice seems difficult.

#### 4. CONTACT AND RAY–SINGER ANALYTIC TORSIONS OF CR SEIFERT MANIFOLDS

We follow [7] to review the definition of a CR Seifert manifold and to fix notation. Note that in dimension 3 a calibrated almost complex structure  $J$  for the contact structure  $H$  is automatically integrable; the pair  $(H, J)$  is often called a *pseudoconvex CR structure*.

**Definition 4.1.** A *CR Seifert manifold* is a 3-dimensional compact manifold  $M$  endowed with a pseudoconvex CR structure  $(H, J)$  and a Seifert structure  $\varphi : \mathbb{S}^1 \rightarrow \operatorname{Diff}(M)$  that are compatible in the following sense: the circle action  $\varphi$  preserves the CR structure and is generated by a Reeb field  $T$ .

It is easily proved that existence of a Reeb field  $T$  satisfying  $\varphi_*(d/dt) = T$  is equivalent to existence of a locally free action of  $\mathbb{S}^1$  whose (never vanishing) infinitesimal generator preserves  $(H, J)$  and is transverse everywhere to  $H$ .

The quotient space  $\Sigma = M/\mathbb{S}^1$  is an orbifold surface with conical singularities. Each CR Seifert manifold is then the  $\mathbb{S}^1$ -bundle inside a line orbifold bundle  $L$  over the compact Riemannian orbifold  $\Sigma$ . Singularities of  $L$  are located above the singularities of  $\Sigma$  in such a way that the total space  $M$  of the bundle is a smooth manifold: if the local fundamental group at  $p \in \Sigma$  is  $\mathbb{Z}/\alpha\mathbb{Z}$  ( $\alpha \in \mathbb{N}^*$ ), a generator acts on a local chart around  $p$  as  $e^{i\frac{2\pi}{\alpha}}$  and on the fibre above  $p$  as  $e^{i\frac{2\pi\beta}{\alpha}}$ , where  $\alpha$  and  $\beta$  are relatively prime integers with  $1 \leq \alpha < \beta$ .

**Theorem 4.2.** *Let  $M$  be CR Seifert manifold and  $\rho : \pi_1(M) \rightarrow U(N)$  a unitary representation. Then:*

- *The analytic torsion  $T_C(\rho)$  of the twisted contact complex and Ray-Singer analytic torsion  $T_{RS}(\rho)$  satisfy*

$$T_C(\rho) = (T_{RS}(\rho))^{-1}.$$

- *The two Ray-Singer metrics on  $\det H^*(M, \rho)$ , corresponding to the de Rham and contact complexes (see (18) and (19)), coincide, i.e.*

$$\| \cdot \|_C = \| \cdot \|_{RS},$$

*via the isomorphism  $\det H^*(\mathcal{E}_\rho^*, d_H) \cong \det H^*(\Omega_\rho^* M, d)$  coming from Proposition 2.2.*

The remainder of this section will be devoted to the proof of these results.

We first need to compare the two spectra coming from the de Rham and contact complexes. This has been done in [7, §§7, 8] in the untwisted case, i.e. for a trivial representation. We will rely on and refer to the spectral analysis done there and point out the few differences coming from the use of the representation  $\rho$  here.

**4.1. Circle action and Fourier analysis.** Let  $V$  be the flat complex vector bundle over  $M$  associated to  $\rho$ . It is the quotient of the trivial bundle  $\widetilde{M} \times \mathbb{C}^N$  over the universal cover  $\widetilde{M}$  of  $M$  by the deck transformations  $\gamma.(m, v) = (\tau(\gamma)m, \rho(\gamma)v)$ . In what follows the contact complex is twisted by  $\rho$  in order to take values in  $V$ .

Let  $\varphi_t$  be the circle action on  $M$  induced by the Reeb field  $T$ . It may be lifted on  $V$ , by parallel transport for the flat connection  $\nabla_\rho$ , but no longer as a circle action. We have instead

$$\varphi_{2\pi} = \rho(f),$$

where  $f = \varphi_{[0, 2\pi]}(m)$  is the generic closed orbit of the action, as seen in  $\pi_1(M)$ . This  $f$  is central, as comes from the presentation of the fundamental group of Seifert manifolds.

**Proposition 4.3** (see e.g. [20, 29, 39]). *Let  $M$  be the circle  $V$ -bundle  $L$  of rational degree  $d = b + \sum_i \frac{\beta_i}{\alpha_i}$  over the orbifold surface  $\Sigma$  of integral genus  $g$ , with  $n$  conical points  $x_i$  of type  $(\alpha_i, \beta_i)$ . Then  $\pi_1(M)$  admits the presentation*

$$\begin{aligned} \pi_1(M) &= \langle f, a_j, b_j \ (1 \leq j \leq g), g_i \ (1 \leq i \leq n) \mid \\ &[a_j, f] = [b_j, f] = [g_i, f] = g_i^{\alpha_i} f^{-\beta_i} = f^b \prod_j [a_j, b_j] \prod_i g_i = 1 \rangle. \end{aligned}$$

We can split  $V$  into irreducible components, where we have

$$\rho(f) = e^{2i\pi x}$$

for some  $x \in [0, 1[$ . We recover a circle action on each such component  $V^x$  by setting

$$\psi_t = e^{-itx} \varphi_t.$$

Using the circle action  $\psi_t$  one can still perform a Fourier decomposition as in [7, §7]. Namely, for  $f \in V^x$ , one has

$$f = \sum_{n \in \mathbb{Z}} \pi_n f \quad \text{where} \quad \pi_n f = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} \psi_t(f) dt$$

satisfies  $\psi_t(\pi_n f) = e^{int} \pi_n f$  and  $T(\pi_n f) = i(n+x)\pi_n f$ . Hence the spectrum of  $iT$  becomes the shifted  $\mathbb{Z} - x$  on  $V^x$ . For  $\lambda = -n - x$  we note

$$(30) \quad V_\lambda = V_n^x = V^x \cap \{iT = -n - x\} = \pi_n(V^x).$$

As the circle action preserves the metric and the whole pseudohermitian structure  $(H, \theta, J)$ , we can split both the Hodge-de Rham and the contact complex spectra into their Fourier components. This is useful for comparing the spectra.

**4.2. Comparing the Riemannian and sub-Riemannian spectra.** Following Propositions 7.2 and 7.4 in [7], we consider the following spaces.

**Definition 4.4.** • Let  $\mathcal{H}_V^2$  be the space of vertical 2-forms  $\alpha = \theta \wedge \beta$ , with values in  $V$ , such that both  $\alpha$  and  $J\alpha$  are closed.

• Let also  $\mathcal{H}_V^0$  be the space of pluri-CR functions in  $V$ , i.e.

$$\mathcal{H}_V^0 = \ker(\Delta_H - T^2) = \ker \bar{\square}_V \square_V$$

with  $\square_V = \partial_V^* \partial_V$  and  $\bar{\square}_V = \bar{\partial}_V^* \bar{\partial}_V$ .

According to [7, (63)–(68)] the non-zero spectrum of the non-positive second-order Laplacian  $P = D^* + \delta_H d_H$  on 2-forms splits into

$$(31) \quad \begin{aligned} \text{spec}^*(P) &= \text{spec}^*(D^*) \cup \text{spec}^*(\Delta_H) \\ &= \text{spec}^*(\Delta_H) \cup \text{spec}^*(-\Delta_H) \\ &\quad \bigcup \text{spec}^*(-JT \mid \mathcal{H}_V^2) \setminus \text{spec}^*(-|T| \mid \mathcal{H}_V^0), \end{aligned}$$

where  $\Delta_H = d_H \delta_H + \delta_H d_H$  is the horizontal second-order Laplacian on functions.

The torsion function  $\kappa_C$  of the contact complex is defined using the fourth-order Laplacians  $\Delta_H^2$  on functions, and  $\Delta_1 = D^* D + (d_H \delta_H)^2$  on 1-forms, with  $\Delta_1$  conjugated to  $P^2$  by Hodge  $*$  duality. Hence (31) yields

$$(32) \quad \text{spec}^*(\Delta_1) = 2 \times \text{spec}^*(\Delta_H^2) \bigcup \text{spec}^*(-T^2 \mid \mathcal{H}_V^2) \setminus \text{spec}^*(-T^2 \mid \mathcal{H}_V^0).$$

Finally by (20) the torsion function of the contact complex reads

$$(33) \quad \begin{aligned} \kappa_C(s) &= 2\zeta(\Delta_H^2)(s) - \zeta(\Delta_1)(s) \\ &= \zeta^*(-T^2 \mid \mathcal{H}_V^0)(s) - \zeta^*(-T^2 \mid \mathcal{H}_V^2)(s) + \kappa(M, \rho), \end{aligned}$$

where we have set

$$(34) \quad \begin{aligned} \kappa(M, \rho) &= 2 \dim(\ker \Delta_H) - \dim(\ker \Delta_1) \\ &= 2 \dim(H^0(M, \rho)) - \dim(H^1(M, \rho)), \end{aligned}$$

since the twisted contact complex is a resolution computing the cohomology of  $M$  with values in  $V$ .

We proceed similarly for the Hodge–de Rham spectrum, and work again with the calibrated metric  $g = d\theta(\cdot, J\cdot) + \theta^2$ . Set

$$(35) \quad Q^\pm = \pm \frac{1}{2} + \sqrt{\frac{1}{4} + \Delta_0}$$

where  $\Delta_0$  is Hodge–de Rham Laplacian acting on functions. According to [7, Corollary 7.6] the spectrum of  $d^*$  on 2-forms splits as

$$(36) \quad \text{spec}^*(d^*) = \text{spec}^*(Q^+) \bigcup \text{spec}^*(-Q^- \mid (\mathcal{H}_V^0)^\perp) \bigcup \text{spec}^*(-JT \mid \mathcal{H}_V^2).$$

By Definition 4.4, we have  $\Delta_0 = \Delta_H - T^2 = |T| - T^2$  on  $\mathcal{H}_V^0$  so that

$$Q^- = -1/2 + \sqrt{1/4 + \Delta_0} = |T| \quad \text{on} \quad \mathcal{H}_V^0.$$

Then (36) also reads

$$\text{spec}^*(d^*) = \text{spec}^*(Q^+) \bigcup \text{spec}^*(-Q^-) \setminus \text{spec}^*(-|T| \mid \mathcal{H}_V^0) \bigcup \text{spec}^*(-JT \mid \mathcal{H}_V^2),$$

and since  $\delta d$  on 1-forms is  $*$  conjugated to  $(d^*)^2$  on 2-forms, we get

$$(37) \quad \text{spec}^*(\delta d) = \text{spec}^*(Q^+)^2 \bigcup \text{spec}^*(Q^-)^2 \bigcup \text{spec}^*(-T^2 \mid \mathcal{H}_V^2) \setminus \text{spec}^*(-T^2 \mid \mathcal{H}_V^0).$$

Now following our convention on analytic torsion, inverse to the original definition of Ray–Singer [33] (see Remark 2.5 and (17)), the torsion function of de Rham complex reads in dimension 3 as

$$\begin{aligned} \kappa_{RS}(s) &= \sum_{k=0}^3 (-1)^{k+1} k \zeta(\Delta_k)(s) \\ &= 3\zeta(\Delta_0)(s) - \zeta(\Delta_1)(s), \end{aligned}$$

with  $\Delta_i = d\delta + \delta d$  the Hodge–de Rham Laplacians on  $i$ -forms. Using

$$\zeta^*(\Delta_1)(s) = \zeta^*(\delta d)(s) + \zeta^*(\Delta_0)(s)$$

and (37) one finds that

$$\begin{aligned} \kappa_{RS}(s) &= 2\zeta^*(\Delta_0)(s) - \zeta^*(\delta d)(s) + 3 \dim(\ker \Delta_0) - \dim(\ker \Delta_1) \\ &= 2\zeta(\Delta_0)(s) - \zeta(Q^+)(2s) - \zeta(Q^-)(2s) \\ &\quad - \zeta^*(-T^2 \mid \mathcal{H}_V^2)(s) + \zeta^*(-T^2 \mid \mathcal{H}_V^0)(s) + \kappa(M, \rho), \end{aligned}$$

since  $\ker Q^+ = \{0\}$  and  $\ker Q^- = \ker \Delta_0$  by the definition (35) of  $Q^\pm$ . Comparing to the contact-complex torsion (34) we have shown the following result.

**Proposition 4.5.** *On a CR Seifert manifold, the Ray–Singer and contact complex torsion functions twisted by a unitary representation satisfy*

$$(38) \quad \kappa_{RS}(s) - \kappa_C(s) = 2\zeta(\Delta_0)(s) - \zeta(Q^+)(2s) - \zeta(Q^-)(2s),$$

where  $\Delta_0$  is the Hodge–de Rham Laplacian on functions and  $Q^\pm = \pm 1/2 + \sqrt{1/4 + \Delta_0}$ .

Note at this stage that the right-hand side of (38) vanishes at  $s = 0$ , as needed by the vanishing of both torsion functions at  $s = 0$ ; see [33] and Corollary 3.7. This also follows from  $\zeta(\Delta_0)(0) = 0$ , for the Hodge–de Rham Laplacian in odd dimension, and that  $\zeta(Q^+)(0) = -\zeta(Q^-)(0)$ , by [7, Lemma 8.5].

**4.3. Proof of Theorem 4.2.** In view of Proposition 4.5 we need to show that if we set

$$Q(s) = \zeta(Q^+)(s) + \zeta(Q^-)(s) - 2\zeta(\Delta)(s/2),$$

writing  $\Delta$  instead of  $\Delta_0$  (the Hodge–de Rham Laplacian on functions) for brevity, then  $Q'(0) = 0$ . We have a hint that this is true by examining finite energy cut-offs: at any finite spectral level ( $\Delta \leq \lambda$ ) it holds that

$$\begin{aligned} \zeta(Q^+)'(0) + \zeta(Q^-)'(0) &= -\ln \det(Q^+) - \ln \det(Q^-) \\ &= -\ln \det(Q^+ \times Q^-) \\ &= -\ln \det \Delta \\ &= \zeta(\Delta)'(0). \end{aligned}$$

Hence  $Q'(0)$  is insensitive to finite eigenvalues and behaves like a pseudodifferential invariant. It may indeed be seen as a multiplicative anomaly for the regularized determinant of the product of the two commuting operators  $Q^\pm$ . As thus  $Q'(0)$  is related to a Wodzicky residue-type invariant; see [24, §6.5].

In fact the spectral function  $Q$  makes sense on any compact Riemannian manifold. The following result holds in a far more general setting than CR Seifert manifolds.

**Lemma 4.6.** *On any odd-dimensional compact Riemannian manifold  $Q(0) = Q'(0) = 0$ .*

*Proof.* Consider the one-parameter deformation

$$Q_\lambda^\pm = \pm \lambda + \sqrt{\lambda^2 + \Delta},$$

so that, with  $\lambda = 1/2$ ,  $Q_{1/2}^\pm$  coincides with our original  $Q^\pm$ . Note that the product formula

$$Q_\lambda^+ \times Q_\lambda^- = \Delta$$

we already mentioned is preserved during the deformation. By ellipticity of  $Q_\lambda^\pm$  and a Mellin transform

$$\zeta^*(Q_\lambda^\pm)(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} (\text{Tr}(e^{-tQ_\lambda^\pm}) - \dim \ker Q_\lambda^\pm) dt$$

is holomorphic for large  $s$ . Define a function, holomorphic for large  $s$ ,

$$F^\pm(\lambda, s) = \int_0^{+\infty} t^{s-1} \text{Tr}^*(e^{-tQ_\lambda^\pm}) dt,$$

where here and in the sequel  $\text{Tr}^*(P) = \text{Tr}(P) - P(\text{const. function} = 1)$ . In particular

$$\begin{aligned}\zeta(Q_{1/2}^+)(s) &= \zeta^*(Q_{1/2}^+)(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \text{Tr}(e^{-tQ_{1/2}^+}) dt \\ &= 1 + \Gamma(s)^{-1} F^+(1/2, s),\end{aligned}$$

and

$$\begin{aligned}\zeta(Q_{1/2}^-)(s) &= 1 + \zeta^*(Q_{1/2}^-)(s) = 1 + \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \text{Tr}(e^{-tQ_{1/2}^-} - 1) dt \\ &= 1 + \Gamma(s)^{-1} F^-(1/2, s).\end{aligned}$$

Thus

$$\begin{aligned}Q(s) &= 1 + \Gamma(s)^{-1} F^+(1/2, s) + 1 + \Gamma(s)^{-1} F^-(1/2, s) - 2 - 2\Gamma(s)^{-1} F(0, s) \\ &\sim s(F^+(1/2, s) + F^-(1/2, s) - 2F(0, s))\end{aligned}$$

when  $s \rightarrow 0$ , with  $F(0, s) = F^+(0, s) = F^-(0, s)$ . Therefore we need to show that

$$F^+(1/2, 0) + F^-(1/2, 0) - 2F(0, 0) = 0,$$

for which it clearly suffices to show that

$$(39) \quad \partial_\lambda F^+(\lambda, 0) + \partial_\lambda F^-(\lambda, 0) = 0.$$

Now one has, for the smooth family of commuting elliptic first-order operators  $Q_\lambda$ ,

$$\begin{aligned}\frac{d}{d\lambda}(e^{-tQ_\lambda^\pm}) &= -t\left(\pm 1 + \frac{\lambda}{\sqrt{\lambda^2 + \Delta}}\right)e^{-tQ_\lambda^\pm} \\ &= \frac{\pm t}{\sqrt{\lambda^2 + \Delta}} \frac{d}{dt}(e^{-tQ_\lambda^\pm}),\end{aligned}$$

so that

$$\partial_\lambda F^+(\lambda, s) + \partial_\lambda F^-(\lambda, s) = \int_0^{+\infty} t^s \frac{d}{dt} \text{Tr}^*\left(\frac{e^{-tQ_\lambda^+} - e^{-tQ_\lambda^-}}{\sqrt{\lambda^2 + \Delta}}\right) dt$$

or after integrating by parts,

$$\begin{aligned}&= s \int_0^{+\infty} t^{s-1} \text{Tr}^*\left(\frac{e^{-tQ_\lambda^+} - e^{-tQ_\lambda^-}}{\sqrt{\lambda^2 + \Delta}}\right) dt \\ &= s \int_0^{+\infty} t^{s-1} 2 \sinh(t\lambda) \text{Tr}^*\left(\frac{e^{-t\sqrt{\lambda^2 + \Delta}}}{\sqrt{\lambda^2 + \Delta}}\right) dt,\end{aligned}$$

or after again integrating by parts,

$$(40) \quad \partial_\lambda F^+(\lambda, s) + \partial_\lambda F^-(\lambda, s) = 2s \int_0^{+\infty} g(\lambda, t, s) \text{Tr}^*(e^{-t\sqrt{\lambda^2 + \Delta}}) dt$$

with

$$g(\lambda, t, s) = \int_0^t u^{s-1} \sinh(u\lambda) du.$$

We therefore need to study the residue at  $s = 0$  of the integral expression in (40). First  $g$  is easily expanded as

$$(41) \quad \begin{aligned} g(\lambda, t, s) &= \int_0^t u^{s-1} \sum_{p \geq 0} \frac{\lambda^{2p+1} u^{2p+1}}{(2p+1)!} du \\ &= \sum_{p \geq 0} \frac{\lambda^{2p+1} t^{2p+1+s}}{(2p+1)!(2p+s+1)}. \end{aligned}$$

Consider next the Poisson kernel  $\text{Tr}^*(e^{-t\sqrt{\lambda^2+\Delta}})$  in (40); the beginning of its asymptotic expansion as  $t \searrow 0$  is related to that of the heat kernel  $\text{Tr}(e^{-t(\lambda^2+\Delta)})$  as follows. Recall from e.g. [22, Lemma 1.7.4] that the trace of the heat kernel of a second-order elliptic Laplacian such as  $P = \lambda^2 + \Delta$  develops when  $t \searrow 0$  as

$$\text{Tr}(e^{-tP}) \sim \sum_{k \geq 0} c_k t^{k-\frac{m}{2}},$$

where  $m$  is the manifold dimension and the  $c_k$  are integrals of curvature terms.

**Proposition 4.7.** *One has for  $P$  and  $c_k$  as above, for odd and even dimension  $m$ ,*

$$(42) \quad \text{Tr}^*(e^{-tP^{1/2}}) = \sum_{k=0}^{[m/2]} \frac{2^{m-2k}}{\sqrt{\pi}} \Gamma\left(\frac{m-2k+1}{2}\right) c_k t^{2k-m} - 1 + f(t)$$

where  $f(t) \rightarrow 0$  when  $t \searrow 0$ .

*Proof.* This is a particular case of [1, Theorem 3.1]. Indeed Bär and Moroianu gave there the full development of such kernels on the diagonal. Higher order terms in the development of the Poisson kernel are more involved in odd dimension since they contain log and even non-local coefficients.

Here is an alternative proof of the partial development we need. The classical Laplace transform  $\mathcal{L}(t^{-1/2}e^{-1/t}) = \sqrt{\pi}p^{-1/2}e^{-2p^{1/2}}$  leads to the subordination formula

$$e^{-tP^{1/2}} = \pi^{-1/2} \int_0^{+\infty} e^{-u} u^{-1/2} e^{-t^2 P/4u} du$$

between Poisson and heat kernels. Therefore summing at the trace level,

$$\begin{aligned} \text{Tr}(e^{-tP^{1/2}}) &= \pi^{-1/2} \int_0^{+\infty} e^{-u} u^{-1/2} \text{Tr}(e^{-t^2 P/4u}) du \\ &= \pi^{-1/2} \int_0^{+\infty} e^{-u} u^{-1/2} \left( \sum_{k=0}^{[m/2]} c_k t^{2k-m} (4u)^{m/2-k} + B(t^2/4u) \right) du \\ &= \sum_{k=0}^{[m/2]} \frac{2^{m-2k}}{\sqrt{\pi}} \Gamma\left(\frac{m-2k+1}{2}\right) c_k t^{2k-m} + \pi^{-1/2} \int_0^{+\infty} e^{-u} u^{-1/2} B\left(\frac{t^2}{4u}\right) du \end{aligned}$$

with  $B(t^2/4u)$  bounded and  $B(v) \rightarrow 0$  when  $v \searrow 0$ . This gives (42) by dominated convergence and the remark that

$$\text{Tr}^*(e^{-tP^{1/2}}) = \text{Tr}(e^{-tP^{1/2}}) - e^{-t\lambda} = \text{Tr}(e^{-tP^{1/2}}) - 1 + o(1).$$



□

We can now complete the proof of Lemma 4.6. We split (40) into

$$\partial_\lambda F^+(\lambda, s) + \partial_\lambda F^-(\lambda, s) = 2s \left( \int_0^1 + \int_1^{+\infty} \right) g(\lambda, t, s) \operatorname{Tr}^*(e^{-tP^{1/2}}) dt.$$

By (41) the second integral here is meromorphic with simple poles at  $s = -2n - 1$  for  $n \in \mathbb{N}$ ; in particular it is regular at  $s = 0$ . Set

$$c'_k = \frac{2^{m-2k}}{\sqrt{\pi}} \Gamma\left(\frac{m-2k+1}{2}\right) c_k,$$

so that by (42)

$$\begin{aligned} \int_0^1 g(\lambda, u, s) \operatorname{Tr}^*(e^{-uP^{1/2}}) du &= \int_0^1 \sum_{0 \leq k \leq [m/2]} \sum_{p \geq 0} \frac{\lambda^{2p+1} u^{2p+1+s}}{(2p+1)!(2p+1+s)} c'_k u^{2k-m} du \\ &\quad + \int_0^1 g(\lambda, u, s) (f(u) - 1) du \\ &= \sum_{0 \leq k \leq [m/2]} \sum_{p \geq 0} \frac{c'_k \lambda^{2p+1}}{(2p+1)!(2p+1+s)(2p+2+2k-m+s)} \\ &\quad + \text{holomorphic terms for } \operatorname{Re}(s) > -1. \end{aligned}$$

This expression has no pole at  $s = 0$  if  $m$  is odd, giving (39) and hence Lemma 4.6. □

We now prove Theorem 4.2. First Proposition 4.5, Lemma 4.6 and (17) show that

$$T_C(\rho) = \exp(\kappa'_C(0)/2) = \exp(\kappa'_{RS}(0)/2) = (T_{RS}(\rho))^{-1}.$$

The equality of Ray–Singer metrics now comes from (18) and (19), using the equality of  $L^2$  metrics on  $H^*(M, \rho)$  when  $H^*(M, \rho)$  is represented by harmonic forms in the de Rham and contact complexes. Indeed these latter two notions coincide on CR Seifert manifolds because of vanishing Tanaka–Webster torsion; see [37, Proposition 12].

## 5. THE TORSION FUNCTION OF CR SEIFERT MANIFOLDS AND ITS DYNAMICAL ASPECTS

As an illustration of our viewpoint on analytic torsion, we first show how to compute it on any CR Seifert 3-manifold  $M$  equipped with a unitary representation  $\rho : \pi_1(M) \rightarrow U(N)$ .

As we will be only concerned with the contact torsion function  $\kappa_C$  in the sequel, we will denote it by  $\kappa$  for brevity.

Surprisingly, on CR Seifert manifolds, we will see that the *whole* contact torsion function  $\kappa$ , not only  $\kappa'(0)$ , is expressible using topological data and combinations of Riemann–Hurwitz zeta functions, parametrised by dynamical properties of  $\rho$  with respect to the circle action on  $M$ . This leads in particular to a Lefschetz-type formula for the torsion; see Theorem 5.7. This extends a result obtained by Fried [17] in the acyclic case using topological methods.

In fact it turns out that our spectral torsion function  $\kappa(s)$  may also be seen as a purely dynamical zeta function, constructed from holonomies along all closed orbits of the Reeb field  $T$  and its length spectrum. This will be shown in §§5.3 and 5.4.

**5.1. Torsion function  $\kappa$  and the Riemann–Roch–Kawasaki formula.** We first relate  $\kappa$  to basic holomorphic data.

Let  $\overline{V}$  be the conjugate complex vector space to  $V$ , i.e. the same underlying real space with the opposite complex structure, and set

$$W = V \oplus \overline{V}.$$

Consider the  $\overline{\partial}_W$  complex

$$(43) \quad \overline{\partial}_W : \Omega^0 W = \Gamma(M, W) \longrightarrow \Omega^{0,1} W = \Omega^{0,1} H \otimes W = \Gamma(M, \Lambda^{0,1} H^* \otimes W)$$

given on the CR (holomorphic) bundle  $W$  and let

$$(44) \quad \mathcal{H}_W^0 = \ker \overline{\partial}_W \quad \text{and} \quad \mathcal{H}_W^1 = \ker \overline{\partial}_W^*$$

denote its cohomology. Using  $\square_{V_\rho} - \square_{V_\rho} = iT$ , see e.g. [7, (57)], one gets

$$\text{spec}(-|T| \mid \mathcal{H}_V^0) = \text{spec}(iT \mid \mathcal{H}_W^0),$$

while, using Hodge  $*$  duality,

$$\text{spec}(-JT \mid \mathcal{H}_V^2) = \text{spec}(iT \mid \mathcal{H}_W^1).$$

Then the spectral decomposition (31) reads as follows.

**Proposition 5.1.** *The spectrum of  $P = D * + \delta_H d_H$  splits as*

$$\begin{aligned} \text{spec}^*(P) &= \text{spec}^*(\Delta_H) \cup \text{spec}^*(-\Delta_H) \\ &\quad \bigcup \text{spec}^*(iT \mid \mathcal{H}_W^1) \setminus \text{spec}^*(iT \mid \mathcal{H}_W^0), \end{aligned}$$

where  $W = V \oplus \overline{V}$  and  $\mathcal{H}_W^*$  is the cohomology of  $\overline{\partial}_W$  as in (43)–(44).

*Remark 5.2.* Compared to the trivial representation case treated in [7, (68)], the only change here is the tensorisation by  $W$ .

Now using the  $\overline{\partial}_W$  complex, equation (32) yields

$$\text{spec}^*(\Delta_1) = 2 \times \text{spec}^*(\Delta_H^2) \bigcup \text{spec}^*(-T^2 \mid \mathcal{H}_W^1) \setminus \text{spec}^*(-T^2 \mid \mathcal{H}_W^0),$$

and (33) becomes

$$(45) \quad \begin{aligned} \kappa(s) &= 2\zeta(\Delta_H^2)(s) - \zeta(\Delta_1)(s) \\ &= \zeta^*(-T^2 \mid \mathcal{H}_W^1)(s) - \zeta^*(-T^2 \mid \mathcal{H}_W^0)(s) + \kappa(M, \rho). \end{aligned}$$

This Lefschetz-type formula for  $\kappa$  can be seen more topologically. Indeed, Fourier decompose each  $V^x$  and let

$$W = \bigoplus_{\lambda \in \text{spec}(iT)} W_\lambda \quad \text{with} \quad W_\lambda = V_\lambda \oplus \overline{V}_\lambda.$$

Then using the holomorphic genus

$$\chi_{\overline{\partial}}(W_\lambda) = \dim \mathcal{H}_{W_\lambda}^1 - \dim \mathcal{H}_{W_\lambda}^0,$$

the torsion function also reads as the Dirichlet series

$$(46) \quad \kappa(s) = \sum_{\lambda \in \text{spec}^*(iT)} \frac{\chi_{\overline{\partial}}(W_\lambda)}{\lambda^{2s}} + \kappa(M, \rho),$$

where, from (30),  $\text{spec}(iT)$  splits into copies of  $-(\mathbb{Z} + x)$  on each sub-representation  $V^x$  of  $V$ , on which  $\rho(f) = e^{2i\pi x}$ .

*Remark 5.3.* For comparison, the eta function of  $P = D * + \delta_H d_H$  twisted by  $\rho$  may be expressed using Proposition 5.1 in a similar manner. One gets

$$(47) \quad \eta_\rho(P)(s) = \sum_{\lambda \in \text{spec}^*(iT)} \text{sign}(\lambda) \frac{\chi_{\bar{\partial}}(V_\lambda) - \chi_{\bar{\partial}}(\overline{V_\lambda})}{|\lambda|^s},$$

which is strikingly the ‘odd version’ of the formula (46) for the torsion function  $\kappa$ . Note that by [7, Theorem 8.8],  $\eta_\rho(P)(0)$  identifies with  $\eta_0(M, \rho)$ , the adiabatic limit of the Riemannian eta invariant with value in  $\rho$ , i.e., the constant term in the development of  $\eta(M, g_\varepsilon, \rho)$  for the adiabatic metrics (which we also consider in the present paper)  $g_\varepsilon = \varepsilon^{-1}d\theta + \varepsilon^{-2}\theta^2$ .

In order to express the series (46) using the Riemann–Roch–Kawasaki formula, we need first to see the bundles  $V_\lambda$ , a priori defined over  $M$ , as  $V$ -bundles over the orbifold  $\Sigma$ , and compute their degrees and orbifold exponents.

Recall that we consider in (30) the circle action given by

$$\psi_t = e^{-itx} \varphi_t = e^{int} \quad \text{on} \quad V_\lambda = V_n^x,$$

where  $\varphi_t$  acts by parallel transport along  $T$  in the flat  $V$ . Now  $e^{it}$  is a local coordinate on the fibre of the circle  $V$ -bundle  $M = S(L) \rightarrow \Sigma$ , hence  $e^{int}$  is a coordinate for  $S(L^n)$ . Thus we have

$$(48) \quad V_\lambda = S(L^{-n}) \otimes V_0^x.$$

Sections of  $V_0^x$  are invariant by the circle action  $\psi_t$  and satisfy  $\varphi_t = e^{itx}$ . Then one can see  $V_0^x$  as defined over  $\Sigma$ , but equipped with the non-flat connection

$$\nabla_x^\Sigma = \nabla_\rho^{\text{flat}} - ix\theta,$$

for which sections  $s$  of  $V_0^x$  are parallel along  $T$ .

Therefore, as seen from  $\Sigma$ ,  $(V_0^x, \nabla^\Sigma)$  has curvature  $\Omega = -ixd\theta$  and rational degree (see [7, 20, 29])

$$\begin{aligned} d(V_0^x) &= \frac{i}{2\pi} \int_\Sigma \text{Tr}_{V_0^x}(-ixd\theta) \\ &= -\dim(V^x)xd(L), \end{aligned}$$

because  $\Omega_L = id\theta$ . Recall that by Proposition 4.3,  $d(L) = b + \sum_i \frac{\beta_i}{\alpha_i}$ . Hence by (48) we get

$$(49) \quad d(V_\lambda) = \dim(V^x)(-nd(L) - xd(L)) = \dim(V^x)\lambda d(L).$$

By conjugation we have also

$$d(\overline{V_\lambda}) = -\dim(V^x)\lambda d(L) = -d(V_\lambda),$$

so that finally

$$d(W_\lambda) = d(\overline{V_{\rho, \lambda}}) + d(V_{\rho, \lambda}) = 0,$$

as expected for this smooth part due to the real structure on  $W_\lambda = V_\lambda \oplus \overline{V_\lambda}$ ; see [26, §14]. We did the above computation for completeness, as the degree of  $V_\lambda$  is needed to study the eta function given in Remark 5.3.

We determine now the local action on  $V_\lambda$  of  $\Gamma_i = \mathbb{Z}/\alpha_i\mathbb{Z}$  around an orbifold point  $x_i \in \Sigma$ . Recall that locally over  $x_i$ ,  $M = S(L)$  is the quotient of  $\mathbb{C} \times S^1$  by the action of  $\Gamma_i$  generated by

$$g \cdot (z_1, z_2) = (e^{2i\pi/\alpha_i} z_1, e^{2i\pi\beta_i/\alpha_i} z_2).$$

Since  $z_2 = e^{it}$  here, we get the local action on  $V_0^x$

$$g \cdot (z_1, z_V) = (e^{2i\pi/\alpha_i} z_1, \psi(2\pi\beta_i/\alpha_i) z_V).$$

Let  $f_i = \varphi([0, 2\pi/\alpha_i])$  be the exceptional closed primitive orbit over  $x_i$ . We have

$$\begin{aligned} \psi(2\pi\beta_i/\alpha_i) &= e^{-2i\pi\beta_i x/\alpha_i} \varphi(2\pi\beta_i/\alpha_i) \\ (50) \quad &= e^{-i2\pi\beta_i x/\alpha_i} \rho(f_i)^{\beta_i}. \end{aligned}$$

As  $f_i^{\alpha_i} = f$  in  $\pi_1(M)$ ,  $\rho(f_i)^{\alpha_i} = \rho(f)$  and the spectra of  $\rho(f_i)$  satisfy

$$(51) \quad \begin{cases} \text{spec } \rho(f_i) = \{e^{2i\pi x_{i,j}} \mid 1 \leq j \leq \dim V_0^x\} \\ \text{with } x_{i,j} = \frac{x + k_{i,j}}{\alpha_i} \in [0, 1) \text{ and } k_{i,j} \in \mathbb{Z}. \end{cases}$$

Hence (50) means that  $\text{spec}(g) = \{e^{2i\pi k_{i,j}\beta_i/\alpha_i}\}$  on the fibres of  $V_0^x$  around  $x_i$ , and, by tensorisation with  $L^{-n}$  in (48), the isotropy exponents of the action on  $V_\lambda$  are all the couples  $(\alpha_i, (-n + k_{i,j})\beta_i \bmod \alpha_i)$ .

Now the Riemann–Roch–Kawasaki formula (see [20, 29, 7]) states that

$$\begin{aligned} \chi_{\overline{\partial}}(W_\lambda) &= \dim(W_\lambda)(1 - g) + \deg(W_\lambda) - \sum_{i,j} \left\{ \frac{\beta_i(W_\lambda)}{\alpha_i(W_\lambda)} \right\} \\ (52) \quad &= \dim(V^x) \chi(\tilde{\Sigma}) - \sum_{i,j} \left\{ \frac{(-n + k_{i,j})\beta_i}{\alpha_i} \right\} + \left\{ \frac{(n - k_{i,j})\beta_i}{\alpha_i} \right\}, \end{aligned}$$

where  $\{a\} = a - [a] \in [0, 1)$  is the fractional part of  $a$  and  $\chi(\tilde{\Sigma}) = 2 - 2g$  is the Euler characteristic of the smooth surface  $\tilde{\Sigma}$  associated to the orbifold  $\Sigma$ ; see e.g. [29]. Observe that (52) does give integers since  $\{a\} + \{-a\}$  is 0 when  $a \in \mathbb{Z}$  and 1 otherwise. Recall also that, to ensure smoothness of the  $V$ -bundle  $M = S(L)$ , the numbers  $\alpha_i$  and  $\beta_i$  are assumed relatively prime, thus giving a free action of  $\mathbb{Z}/\alpha_i\mathbb{Z}$  at orbifold points. Hence the fractional part in (52) simplifies using

$$(53) \quad \delta(n, i, j) = \begin{cases} 1 & \text{if } n - k_{i,j} \in \alpha_i\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$(54) \quad \chi_{\overline{\partial}}(W_\lambda) = \dim(V^x) \chi(\Sigma^*) + \sum_{i,j} \delta(n, i, j),$$

where

$$\chi(\Sigma^*) = 2 - 2g - |I|$$

is the Euler characteristic of the punctured surface  $\Sigma^* = \Sigma \setminus \cup_I \{x_i\}$  at the  $|I|$  orbifold points.

For  $a \in ]0, 1]$  let  $\zeta(s, a) = \sum_{n \in \mathbb{N}} \frac{1}{(n+a)^s}$  be Hurwitz zeta function. We can now express  $\kappa$  as a combination of such functions. This is the first step towards the identification of the torsion function as a dynamical zeta function given in §5.3.

**Theorem 5.4.** *Split  $V$  into irreducible  $V^x$ ; then the torsion function spectrally decomposes as*

$$\kappa(s) = \sum_{V^x} \kappa_x(s)$$

such that :

- On  $V^x$  with  $x \in ]0, 1[$ , i.e.  $\rho(f) = e^{2i\pi x} \neq \text{Id}$ , we have

$$(55) \quad \begin{aligned} \kappa_x(s) = & \dim(V^x) \chi(\Sigma^*) (\zeta(2s, x) + \zeta(2s, 1-x)) \\ & + \sum_{i,j} \frac{1}{\alpha_i^{2s}} (\zeta(2s, x_{i,j}) + \zeta(2s, 1-x_{i,j})). \end{aligned}$$

- On  $V^0 = \ker(\text{Id} - \rho(f))$  let  $V^{0,i} = \ker(\text{Id} - \rho(f_i))$ ; then we have

$$(56) \quad \begin{aligned} \kappa_0(s) = & \kappa(M, \rho) (2\zeta(2s) + 1) + 2\zeta(2s) \sum_i \dim(V^{0,i}) (\alpha_i^{-2s} - 1) \\ & + \sum_{i,j \mid x_{i,j} \neq 0} \frac{1}{\alpha_i^{2s}} (\zeta(2s, x_{i,j}) + \zeta(2s, 1-x_{i,j})). \end{aligned}$$

This relates the torsion function to dynamical properties of the circle action here. Indeed apart from the cohomological term  $\kappa(M, \rho)$ , the expression is clearly built on the holonomy properties of  $\rho$  along the various closed primitive orbits of the flow: the generic orbit  $f$  of the action over  $\Sigma^*$ , and associated holonomy  $\rho(f) = e^{2i\pi x}$  on  $V^x$ , and the exceptional orbits  $f_i$  of holonomy  $\rho(f_i) = \{e^{2i\pi x_{i,j}}\}$ .

*Proof.* • We compute first the contribution of  $V^x$  for  $x \neq 0$ , i.e. when  $\rho(f) = e^{2i\pi x} \neq \text{Id}$ . Here  $iT = \lambda = -n - x \neq 0$  always, and by (46) and (54)

$$\begin{aligned} \kappa_x(s) &= \dim(V^x) \chi(\Sigma^*) \sum_{n \in \mathbb{Z}} \frac{1}{|n+x|^{2s}} + \sum_{i,j} \sum_{k \in \mathbb{Z}} \frac{1}{|k\alpha_i + k_{i,j} + x|^{2s}} \\ &= \dim(V^x) \chi(\Sigma^*) \left( \sum_{n \geq 0} \frac{1}{|n+x|^{2s}} + \sum_{n > 0} \frac{1}{|-n+x|^{2s}} \right) \\ &\quad + \sum_{i,j} \frac{1}{\alpha_i^{2s}} \sum_{k \in \mathbb{Z}} \frac{1}{|k+x_{i,j}|^{2s}}, \end{aligned}$$

by (51). This leads to (55).

• We compute now  $\kappa_0$ , including the cohomological term  $\kappa(M, \rho)$  from (34) and (46), since harmonic forms only appear in  $\ker(iT) \subset V_{\rho,0}$ ; see e.g. [37, Proposition 12]. We have

$$(57) \quad \kappa_0(s) = \dim(V^0)\chi(\Sigma^*) \sum_{n \in \mathbb{Z}^*} \frac{1}{|n|^{2s}} + \sum_{i,j} \sum_{n \in \mathbb{Z}^*} \frac{\delta(n,i,j)}{|n|^{2s}} + \kappa(M, \rho).$$

We recall from (34) that

$$\kappa(M, \rho) = 2 \dim H^0(M, \rho) - \dim H^1(M, \rho)$$

can be computed using contact-harmonic forms on  $M$ . By [37, Proposition 12] contact-harmonic forms are both holomorphic and  $T$ -invariant since the Reeb flow preserves  $J$  here, i.e. Tanaka–Webster torsion vanishes. Then one gets

$$\begin{aligned} \kappa(M, \rho) &= 2\chi_{\bar{\partial}}(W_0) \\ &= \dim(V^0)\chi(\Sigma^*) + \sum_{i,j} \delta(0,i,j) \quad \text{by (54)} \\ (58) \quad \kappa(M, \rho) &= \dim(V^0)\chi(\Sigma^*) + \sum_i \dim(V^{0,i}), \end{aligned}$$

since, by (51) and (53),  $\delta(n,i,j) = 1$  if and only if  $x_{i,j} = 0$ . Then (57) reads

$$\kappa_0(s) = \kappa(M, \rho)(2\zeta(2s) + 1) + \sum_{i,j} \sum_{n \in \mathbb{Z}^*} \frac{\delta(n,i,j)}{|n|^{2s}} - 2\zeta(2s) \sum_i \dim(V^{0,i}).$$

We observe now that if  $x_{i,j} \in ]0, 1[$ ,

$$\begin{aligned} \sum_{n \in \mathbb{Z}^*} \frac{\delta(n,i,j)}{|n|^{2s}} &= \sum_{k \in \mathbb{Z}} \frac{1}{|k\alpha_i + k_{i,j}|^{2s}} \\ &= \frac{1}{\alpha_i^{2s}} (\zeta(2s, x_{i,j}) + \zeta(2s, 1 - x_{i,j})) \quad \text{by (51)}. \end{aligned}$$

On the other hand for  $x_{i,j} = 0$ ,

$$\sum_{n \in \mathbb{Z}^*} \frac{\delta(n,i,j)}{|n|^{2s}} = \sum_{k \in \mathbb{Z}^*} \frac{1}{|k\alpha_i|^{2s}} = \frac{2}{\alpha_i^{2s}} \zeta(2s),$$

as needed in (56). □

The expression of  $\kappa_x$  given in Theorem 5.4 vanishes at  $s = 0$ , as it ought to by Corollary 3.8. Here this follows from the classical result  $\zeta(0, a) = 1/2 - a$ ; see [46, §13] for instance. The following observation will be useful in the sequel.

**Corollary 5.5.** *The torsion function  $\kappa$  has a unique simple pole at  $s = 1/2$  with residue*

$$\text{Res}_{1/2}(\kappa) = \chi(\Sigma) \dim V,$$

where  $\chi(\Sigma) = 2 - 2g + \sum_i (\frac{1}{\alpha_i} - 1)$  denotes the rational Euler class of the orbifold  $\Sigma$ .

*Proof.* One knows (see [46, §13]) that the Hurwitz zeta function  $\zeta(2s, a)$  has a unique simple pole at  $s = 1/2$  with residue  $1/2$ . Then (55) on  $V^x$  yields

$$\text{Res}_{1/2}(\kappa_x) = \dim V^x \chi(\Sigma^*) + \dim V^x \sum_i \frac{1}{\alpha_i} = \dim V^x \chi(\Sigma).$$

On the other hand by (56) in  $V^0$ ,

$$\begin{aligned} \text{Res}_{1/2}(\kappa_0) &= \kappa(M, \rho) + \sum_i \dim(V^{0,i}) \left( \frac{1}{\alpha_i} - 1 \right) + \sum_i \dim((V^{0,i})^\perp) \frac{1}{\alpha_i} \\ &= (\kappa(M, \rho) - \sum_i \dim V^{0,i}) + \dim V^0 \sum_i \frac{1}{\alpha_i} \\ &= \dim V^0 (\chi(\Sigma^*) + \sum_i \frac{1}{\alpha_i}) = \dim V^0 \chi(\Sigma), \end{aligned}$$

by (58). □

*Remark 5.6.* We lastly observe that a similar treatment applies to handle the twisted eta function in Remark 5.3. Indeed by the Riemann–Roch–Kawasaki formula and (49) one has

$$\chi_{\overline{\partial}}(V_\lambda) - \chi_{\overline{\partial}}(\overline{V}_\lambda) = 2 \dim(V_\rho^x) \lambda d(L) + \sum_{i,j} \left\{ \frac{(-n + k_{i,j}) \beta_i}{\alpha_i} \right\} - \left\{ \frac{(n - k_{i,j}) \beta_i}{\alpha_i} \right\},$$

and by (47) the contribution of  $V^x$  to eta is

$$\begin{aligned} \eta_x(P)(s) &= 2 \dim(V^x) d(L) \sum_{\lambda \in \text{spec}^*(iT)} \frac{1}{|\lambda|^{s-1}} \\ &\quad + \sum_{i,j} \sum_{\lambda \in \text{spec}^* iT} \left( 2 \left\{ \frac{(-n + k_{i,j}) \beta_i}{\alpha_i} \right\} - 1 + \delta(n, i, j) \right) \frac{\text{sgn}(\lambda)}{|\lambda|^s}. \end{aligned}$$

The generic smooth contribution may be written as

$$2 \dim(V^x) d(L) \times \begin{cases} 2\zeta(s-1) & \text{if } x = 0 \\ \zeta(s-1, x) + \zeta(s-1, 1-x) & \text{if } x \neq 0, \end{cases}$$

taking value

$$- \dim(V^x) d(L) \left( \frac{1}{6} + x(1-x) \right)$$

at  $s = 0$ ; see [46, §13]. Following Nicolaescu's work [29], the remaining ‘periodic’ eta term can be handled using Dedekind–Rademacher sums and Hurwitz functions; see Proposition 1.10 and Lemma 1.11 in [29] for details.

**5.2. A Lefschetz-type formula for the Ray–Singer metric.** Using Theorem 4.2 we can now compute the Ray–Singer analytic torsion  $T_{RS} = \exp(-\kappa'(0)/2) = (T_C)^{-1}$ , which gives the associated Ray–Singer metric on  $\det H^*(M, \rho)$ ,

$$(59) \quad \| \quad \|_{RS} = (T_{RS})^{-1} | \quad |_{L^2(\Omega^*M)}.$$

In the acyclic case, i.e.  $H^*(M, \rho) = 0$ , Fried [17] has shown that the Reidemeister–Franz torsion, and thus the analytic torsion by the works [15, 28] of Cheeger and Müller, may be nicely expressed ‘à la Lefschetz’ using determinants associated to the generic and exceptional holonomies along the primitive orbits of the circle action. For a general unitary representation  $\rho : \pi_1(M) \rightarrow U(N)$  we obtain:

**Theorem 5.7.** *Let  $\rho(f)^\top$  and  $\rho(f_i)^\top$  denote the restriction of these holonomies to respectively  $(V^0)^\perp$  and  $(V^{0,i})^\perp$  with*

$$V^0 = \ker(\text{Id} - \rho(f)) \quad \text{and} \quad V^{0,i} = \ker(\text{Id} - \rho(f_i)).$$

*Then Ray–Singer analytic torsion  $T_{RS}(M, \rho) = \exp(-\kappa'(0)/2)$  is given by*

$$(60) \quad T_{RS}(M, \rho) = (2\pi)^{\kappa(M, \rho)} |\det(\text{Id} - \rho(f)^\top)|^{\chi(\Sigma^*)} \prod_i \frac{|\det(\text{Id} - \rho(f_i)^\top)|}{\alpha_i^{\dim(V^{0,i})}}.$$

*Proof.* By Lerch’s formula  $\partial_s \zeta(s, x)_{s=0} = \ln \Gamma(x) - \frac{1}{2} \ln(2\pi)$ , see [46, §13], we have

$$\begin{aligned} \partial_s \zeta(0, x) + \partial_s \zeta(0, 1-x) &= \ln(\Gamma(x)\Gamma(1-x)/2\pi) \\ &= -\ln(2 \sin(\pi x)) \quad \text{by the Euler reflection formula,} \\ &= -\ln |1 - e^{2i\pi x}|. \end{aligned}$$

Hence by (55) on  $V^x$ ,

$$-\kappa'_x(0)/2 = \dim(V^x) \chi(\Sigma^*) \ln |1 - e^{2i\pi x}| + \sum_{i,j} \ln |1 - e^{2i\pi x_{i,j}}|,$$

which gives the determinant contribution of  $V^x$  to (60).

By (56) on  $V^0$  and Lerch’s formula again, one finds

$$-\kappa'_0(0)/2 = -2\zeta'(0)\kappa(M, \rho) - \sum_i \dim(V^{0,i}) \ln \alpha_i + \sum_{i,j | x_{i,j} \neq 0} \ln |1 - e^{2i\pi x_{i,j}}|$$

similarly as above. This gives the needed contribution of  $V^0$  to (60).  $\square$

As required, formula (60) coincides with that of Fried [17, p. 198] for acyclic representations. The only new factor in our case is the cohomological term  $(2\pi)^{\kappa(M, \rho)}$ . That the full expression for the torsion is ‘quantised’ here is due to the rigidity of volume in this CR Seifert case. Namely, the size of  $\theta$  is fixed such that the circle action is generated by the Reeb field  $T$  in constant time  $2\pi$ , hence the volume forms  $\text{dvol} = \theta \wedge d\theta$  on  $TM$  and  $d\theta$  on  $H$  are also fixed. Then when the complex structure  $J$  changes on the base  $\Sigma$ , together with the calibrated metric  $g = d\theta(\cdot, J\cdot) + \theta^2$ , harmonic forms representing  $H^*(M, \rho)$  change, but their length in  $\det \mathcal{H}^*(M, \rho)$  does not. Hence the  $L^2$  metric used in (59) is independent of  $J$ , as is the analytic torsion given in (60).



**5.3. The contact torsion function as a dynamical zeta function.** In [16, 17], Fried proposed to express the torsion using the following basic dynamical objects.

For each free homotopical class  $C$  of periodic orbit of the Reeb field  $T$ , let  $\ell(C)$  denote its length and  $\text{ind}(C)$  its Fuller index; see [19] or [16, §4] for an account of these notions.

**Proposition 5.8** ([16, Lemma 5.3], [17]). *The free homotopy classes of closed orbits of  $T$  are the following :*

- (1)  $f^n$  with  $n \in \mathbb{N}^*$ , of length  $2\pi n$  and Fuller index  $\chi(\Sigma)/n$ , where

$$\chi(\Sigma) = 2 - 2g - \sum_i (1 - 1/\alpha_i) = \chi(\Sigma^*) + \sum_i 1/\alpha_i$$

*is the rational Euler class of the quotient orbifold  $\Sigma = M/\langle T \rangle$ ;*

- (2) *the isolated  $f_i^n$  for  $n \notin \alpha_i \mathbb{N}$ , of length  $2\pi n/\alpha_i$  and Fuller index  $1/n$ .*

Fried observed that for acyclic unitary representations one has

$$(61) \quad T_{RS}(M, \rho) = |\exp(Z_F(0))|,$$

where  $Z_F(0)$  stands for the analytic continuation at  $s = 0$  of the dynamical function

$$Z_F(s) = - \sum_C \text{ind}(C) \text{Tr}(\rho(C)) e^{-s\ell(C)}.$$

This can be checked directly from the calculation of torsion in Theorem 5.7 and Proposition 5.8, as in [17, §1]. Such a link between analytic torsion and flow dynamics is not coincidental; it has already been observed in many other geometric situations, see e.g. [16]. In particular it holds for the geodesic flow on hyperbolic manifolds, as proved by Fried in [18], or more generally on locally symmetric spaces of non-positive sectional curvature, as proved by Moscovici and Stanton in [27]. These results are rooted in Selberg's trace formula, expressing heat kernel traces as a sum of traces along closed geodesics.

In our setting, in view of Theorem 5.4, it is quite natural to try to express the contact torsion function  $\kappa(s)$  itself using the same dynamical data as above. Indeed the *whole* spectral function  $\kappa$  may be nicely interpreted ‘à la Selberg’ as a purely dynamical zeta function of the Reeb flow.

**Theorem 5.9.** *Let*

$$f(s) = \Gamma(s) \cos\left(\frac{\pi s}{2}\right).$$

*For  $C$  closed let  ${}^R\text{Tr}(\rho(C))$  denote the real part of the trace of  $\rho(C)$  on  $V_\rho$ . Then*

$$(62) \quad f(s)(\kappa(s/2) - \kappa(M, \rho)) = \sum_C \text{ind}(C) {}^R\text{Tr}(\rho(C)) \ell(C)^s,$$

*where the sum is taken over all free homotopical classes of closed orbits of the Reeb flow.*

We first make some comments about this identity. First, to remove the real part in (62), one could also sum over all orbits  $C$  and  $C^{-1}$ , the latter corresponding to the opposite flow  $-T$ . Indeed the torsion function is not sensitive to a change of  $\theta \mapsto -\theta$ , together with  $J \mapsto -J$ , since  $\kappa$  is defined using real operators  $\Delta_0$  and  $\Delta_1$ .

Also let

$$(63) \quad Z_\rho(s) = \sum_C \text{ind}(C) {}^R\text{Tr}(\rho(C)) \ell(C)^s$$

be the dynamical zeta side of (62). From Proposition 5.8 this converges for  $\text{Re}(s) < 0$ . In contrast the spectral side

$$\kappa^*(s/2) = \kappa(s/2) - \kappa(M, \rho) = 2 \text{Tr}^*(\Delta_0^{-s/2}) - \text{Tr}^*(\Delta_1^{-s/2})$$

is a converging series for  $\text{Re}(s) > 2$ . Hence, when seen as series, the spectral and dynamical sides of (62) *never* converge for the same  $s$ , and the identity only holds through meromorphic continuation. When  $s \rightarrow 0$  we get

$$(64) \quad \lim_{s \rightarrow 0} \left( Z_\rho(s) + \frac{\kappa(M, \rho)}{s} \right) = \kappa'(0)/2 = -\ln(T_{RS}(M, \rho)),$$

and (ln of) the analytic torsion may be seen as a topological regularisation of the *formal* dynamical series

$$(65) \quad -'' Z_\rho(0)'' = - \sum_C \text{ind}(C) {}^R\text{Tr}(\rho(C)).$$

Comparing with (61) and Fried's dynamical function yields

$$''Z_\rho(0)'' = Z_\rho(0) = -\text{Re}(Z_F(0)),$$

in the acyclic case, and the dynamical functions  $Z_F$  and  $Z_\rho$  both provide analytic continuation of the same dynamical series in (65). This series has been interpreted in [16, 17] as being the total Fuller measure of periodic orbits, and has a formal invariance by deformation of the flow, as long as orbit periods stay bounded.

We note also that the trace formula (62) can be written in a more symmetric manner

$$(66) \quad \Gamma(s) \kappa^*(s) = \frac{2^{1-2s}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} - s\right) Z_\rho(2s),$$

as follows from the classical identities (see [46])

$$\Gamma(s) \Gamma\left(s + \frac{1}{2}\right) = 2^{1-2s} \sqrt{\pi} \Gamma(2s) \quad \text{and} \quad \Gamma\left(s + \frac{1}{2}\right) \Gamma\left(-s + \frac{1}{2}\right) = \frac{\pi}{\cos(\pi s)}.$$

This formulation will be useful in §5.4.

As a last comment, we observe that the trace formula (62) is homogeneous in the constant rescaling  $\theta \mapsto K\theta$ . Indeed, the metric here is  $g = d\theta(\cdot, J\cdot) + \theta^2$ , hence  $\ell(C) = \int_C \theta$  changes to  $K\ell(C)$ , while the fourth-order contact Laplacians  $\Delta_i$  are homogeneous and rescale to  $K^{-2}\Delta_i$ . Thus  $\zeta^*(\Delta^{1/2})(s)$  rescales to  $K^s \zeta^*(\Delta^{1/2})(s)$  as needed. Such a property does not hold for the Hodge–de Rham Laplacians.

*Proof of Theorem 5.9.* We will start from the expression of  $\kappa(s)$  by Hurwitz zeta functions as given in Theorem 5.4. First Hurwitz's formula (see [46, §13]) states that for  $\text{Re}(s) < 0$

$$\zeta(s, x) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \left[ \sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^{+\infty} \cos\left(\frac{2\pi x n}{n^{1-s}}\right) + \cos\left(\frac{\pi s}{2}\right) \sum_{n=1}^{+\infty} \sin\left(\frac{2\pi x n}{n^{1-s}}\right) \right],$$

so that using  $f(s)f(1-s) = \pi/2$  gives

$$(67) \quad f(s)(\zeta(s, x) + \zeta(s, 1-x)) = \sum_{n=1}^{+\infty} \operatorname{Re}(e^{2i\pi x n}) \frac{(2\pi n)^s}{n}.$$

Note also the corresponding limit expression when  $x \rightarrow 0^+$  with  $\operatorname{Re}(s) < 0$ :

$$(68) \quad 2f(s)\zeta(s) = \sum_{n=1}^{+\infty} \frac{(2\pi n)^s}{n}.$$

- We study the contribution of  $\kappa_x$  on  $V^x$  with  $x \in (0, 1)$ . By (67), formula (55) yields

$$f(s)\kappa_x(s/2) = \chi(\Sigma^*) \sum_{n \geq 1} \frac{\operatorname{RTr}(\rho(f^n))}{n} (2\pi n)^s + \sum_i \sum_{n \geq 1} \frac{\operatorname{RTr}(\rho(f_i^n))}{n} \left(\frac{2\pi n}{\alpha_i}\right)^s,$$

hence by Proposition 5.8,

$$\begin{aligned} f(s)\kappa_x(s/2) &= \sum_{n \geq 1} \left( \operatorname{ind}(f^n) - \sum_i \frac{1}{\alpha_i} \right) \operatorname{RTr}(\rho(f^n)) \ell(f^n)^s \\ &\quad + \sum_i \sum_{n \notin \alpha_i \mathbb{N}} \operatorname{ind}(f_i^n) \operatorname{RTr}(\rho(f_i^n)) \ell(f_i^n)^s \\ &\quad + \sum_i \sum_{k \geq 1} \frac{\operatorname{RTr}(\rho(f^k))}{k\alpha_i} \ell(f^k)^s \\ &= \sum_C \operatorname{ind}(C) \operatorname{RTr}(\rho(C)) l(C)^s, \end{aligned}$$

as needed in (62).

- We now study  $\kappa_0$  on  $V^0 = \ker(\operatorname{Id} - \rho(f))$ . Recall that  $\operatorname{spec} \rho(f_i) = \{e^{2i\pi x_{i,j}}\}$  and  $V^{0,i} = \ker(\operatorname{Id} - \rho(f_i))$ . By (67)–(68) formula (56) reads

$$(69) \quad f(s)(\kappa_0(s/2) - \kappa(M, \rho)) = \kappa(M, \rho) \sum_{n \geq 1} \frac{(2\pi n)^s}{n} + \sum_i \sum_{n \geq 1} \frac{\operatorname{RTr}_{(V^{0,i})^\perp}(\rho(f_i^n))}{n} \left(\frac{2\pi n}{\alpha_i}\right)^s \\ + \sum_i \sum_{n \geq 1} \dim(V^{0,i}) (\alpha_i^{-s} - 1) \frac{(2\pi n)^s}{n}.$$

By (58), the first series reads

$$\begin{aligned} \kappa(M, \rho) \sum_{n \geq 1} \frac{(2\pi n)^s}{n} &= [\dim(V^0)(\chi(\Sigma) - \sum_i \frac{1}{\alpha_i}) + \sum_i \dim(V_0^i)] \sum_{n \geq 1} \frac{(2\pi n)^s}{n} \\ &= \sum_{n \geq 1} \operatorname{ind}(f^n) \operatorname{RTr}(\rho(f^n)) \ell(f^n)^s - \sum_i \sum_{n \geq 1} \dim(V^0) \frac{(2\pi n)^s}{n\alpha_i} \\ &\quad + \sum_i \sum_{n \geq 1} \dim(V^{0,i}) \frac{(2\pi n)^s}{n}. \end{aligned}$$

Since  $\rho(f_i) = \text{Id}$  on  $V^{0,i}$ , the second series in (69) splits into

$$\begin{aligned} \sum_i \sum_{n \geq 1} \frac{\text{RTr}_{(V^{0,i})^\perp}(\rho(f_i^n))}{n} \left(\frac{2\pi n}{\alpha_i}\right)^s &= \sum_i \sum_{n \geq 1} \frac{\text{RTr}(\rho(f_i^n))}{n} \left(\frac{2\pi n}{\alpha_i}\right)^s - \sum_i \sum_{n \geq 1} \frac{\dim V^{0,i}}{n} \left(\frac{2\pi n}{\alpha_i}\right)^s \\ &= \sum_i \sum_{n \notin \alpha_i \mathbb{N}} \text{ind}(f_i^n) \text{RTr}(\rho(f_i^n)) \ell(f_i^n)^s \\ &\quad + \sum_i \sum_{k \geq 1} \dim V^0 \frac{(2\pi k)^s}{k \alpha_i} - \sum_i \sum_{n \geq 1} \frac{\dim V^{0,i}}{n \alpha_i^s} (2\pi n)^s \end{aligned}$$

since  $\rho(f_i^n) = \rho(f_i^k) = \text{Id}$  on  $V^0$  for  $n = k\alpha_i$ . Therefore after cancellations (69) yields

$$f(s)(\kappa_0(s/2) - \kappa(M, \rho)) = \sum_C \text{ind}(C) \text{RTr}(\rho(C)) \ell(C)^s,$$

as needed.  $\square$

**5.4. The torsion heat trace as a dynamical theta function.** In Theorem 5.9, spectral and dynamical aspects of analytic torsion are compared through zeta functions. One can also work at the level of heat kernels. Consider the heat operators of the fourth-order Laplacians  $\Delta_0$  and  $\Delta_1$  of the contact complex, and set

$$\begin{aligned} (70) \quad \text{Tr}_\kappa(e^{-t\Delta}) &= 2 \text{Tr}(e^{-t\Delta_0}) - \text{Tr}(e^{-t\Delta_1}) \\ &= \text{Tr}_\kappa^*(e^{-t\Delta}) + \kappa(M, \rho). \end{aligned}$$

Recall that for  $\text{Re}(s) > 1$ ,

$$(71) \quad \kappa^*(s) = 2\zeta^*(\Delta_0)(s) - \zeta^*(\Delta_1)(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \text{Tr}_\kappa^*(e^{-t\Delta}) dt.$$

Then the following trace formula holds in our CR Seifert setting.

**Theorem 5.10.** *One has*

$$(72) \quad \text{Tr}_\kappa(e^{-t\Delta}) = \dim V \frac{\sqrt{\pi}\chi(\Sigma)}{\sqrt{t}} + \frac{1}{\sqrt{\pi t}} \sum_C \ell(C) \text{ind}(C) \text{RTr}(\rho(C)) e^{-\ell(C)^2/4t},$$

where  $C$  runs over free homotopical classes of closed orbits of the Reeb flow and  $\chi(\Sigma)$  is the rational Euler class of the quotient orbifold.

Hence the torsion heat trace may be compared with the purely dynamical theta function

$$(73) \quad \vartheta(t) = \frac{1}{\sqrt{\pi t}} \sum_C \ell(C) \text{ind}(C) \text{RTr}(\rho(C)) e^{-\ell(C)^2/4t}.$$

We will first need the following fact on the asymptotic heat development.

**Proposition 5.11.** *As  $t \searrow 0$ , it holds that*

$$\text{Tr}_\kappa(e^{-t\Delta}) = \dim V \frac{\sqrt{\pi}\chi(\Sigma)}{\sqrt{t}} + O(\sqrt{t}).$$

*Proof.* By Corollary 5.5, the torsion function  $\kappa$  has a single simple pole at  $s = 1/2$  with residue  $\chi(\Sigma) \dim V$ . On the other hand we know that, for a fourth-order hypoelliptic Laplacian in dimension 3, as  $t \searrow 0$ ,

$$\mathrm{Tr}_\kappa(e^{-t\Delta}) = \frac{c_1}{t} + \frac{c_{1/2}}{\sqrt{t}} + c_0 + O(\sqrt{t}).$$

Mellin's transform (71) split into  $\int_0^1 + \int_1^{+\infty}$  providing

$$c_1 = \mathrm{Res}_{s=1}(\kappa(s)) = 0, \quad c_{1/2} = \Gamma(1/2) \mathrm{Res}_{1/2}(\kappa) = \sqrt{\pi} \chi(\Sigma) \dim V,$$

and

$$c_0 = \kappa(M, \rho) + \mathrm{Res}_0(\Gamma(s) \kappa^*(s)) = 0,$$

since  $\kappa^*(0) = \kappa(0) - \kappa(M, \rho) = -\kappa(M, \rho)$ . □

We can now prove Theorem 5.10.

*Proof.* One takes Mellin transforms  $\mathcal{M}$  of both sides in (72). For  $\mathrm{Re}(s) < 0$  one finds

$$(74) \quad \mathcal{M}(\vartheta)(s) = \frac{2^{1-2s}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} - s\right) Z_\rho(2s).$$

On the other hand (71) and Proposition 5.11 yield that, for  $\mathrm{Re}(s) > 1/2$ ,

$$\mathcal{M}(\mathrm{Tr}_\kappa^*(e^{-t\Delta}))(s) = \Gamma(s) \kappa^*(s).$$

In order to compare these identities we need first to extend them to a common domain. Indeed set

$$\mathrm{Tr}_0(e^{-t\Delta}) = \mathrm{Tr}_\kappa^*(e^{-t\Delta}) + \left( \kappa(M, \rho) - \dim V \frac{\sqrt{\pi} \chi(\Sigma)}{\sqrt{t}} \right) \chi_{]0,1]}(t);$$

by Proposition 5.11 this has a Mellin transform for  $\mathrm{Re}(s) > -1/2$  and

$$(75) \quad \mathcal{M}(\mathrm{Tr}_0(e^{-t\Delta}))(s) = \Gamma(s) \kappa^*(s) + \frac{\kappa(M, \rho)}{s} + \dim V \frac{\sqrt{\pi} \chi(\Sigma)}{s - 1/2}.$$

Set also

$$\vartheta_0(t) = \vartheta(t) - \left( \kappa(M, \rho) - \dim V \frac{\sqrt{\pi} \chi(\Sigma)}{\sqrt{t}} \right) \chi_{[1,+\infty[}(t),$$

so that (74) yields, for  $\mathrm{Re}(s) < 0$ ,

$$\begin{aligned} \mathcal{M}(\vartheta_0)(s) &= \frac{2^{1-2s}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} - s\right) Z_\rho(2s) + \frac{\kappa(M, \rho)}{s} + \dim V \frac{\sqrt{\pi} \chi(\Sigma)}{s - 1/2} \\ &= \mathcal{M}(\mathrm{Tr}_0(e^{-t\Delta}))(s), \end{aligned}$$

for  $-1/2 < \mathrm{Re}(s) < 0$ , by (66) and (74). Hence by injectivity of the Mellin transform, coming from Fourier injectivity on integrable functions here, one concludes that  $\vartheta_0(t) = \mathrm{Tr}_0(e^{-t\Delta})$ , yielding the trace formula.

The authors thank Patrick Gérard for an enlightening discussion on this proof. □

Formula (72) is a typical Selberg-type trace formula, which holds in many other geometric situations, see [16, p. 57] for instance. Note that it holds here even in *variable* curvature. This may look unusual as Selberg's technique is algebraic and relies on group actions, hence makes sense on uniformised (locally symmetric) manifolds. However we know, by (46) or Theorem 5.7, that on CR Seifert manifolds the torsion function  $\kappa(s)$  is 'topological', meaning independent of the complex structure  $J$  since  $\theta$  is fixed by the circle action. Now except for some cases that fibre over the sphere  $S^2$  with two singular points, all other CR Seifert manifolds can be uniformised, i.e. endowed with a constant curvature metric; see e.g. [5] or [20, Theorems 1.1, 1.2]. By Moser's lemma this can be done with a fixed volume. Thus, except for the special cases mentioned, one could have worked over a uniformised orbifold  $\Sigma$ , where Selberg's technique should also be applicable.

The trace formula (72) has a striking consequence for the small time behaviour of the torsion heat trace  $\mathrm{Tr}_\kappa(e^{-t\Delta})$ . Indeed, from its definition (73) and Proposition 5.8, the dynamical theta function  $\vartheta$  clearly decays very fast as

$$\vartheta(t) = O(e^{-C/t}) \quad \text{when } t \searrow 0,$$

so that instead of Proposition 5.11 we get the *full* heat development.

**Corollary 5.12.** *On CR Seifert manifolds, as  $t \searrow 0$  we have*

$$\mathrm{Tr}_\kappa(e^{-t\Delta}) = \dim V \frac{\sqrt{\pi}\chi(\Sigma)}{\sqrt{t}} + O(e^{-C/t}).$$

Thus on such manifolds this torsion heat trace does show a 'fantastic cancellations' phenomenon as encountered for the heat of Dirac operators.

Note that the only surviving term as  $t \searrow 0$  in this heat development is the integral of curvature data, as should be the case. Gauss–Bonnet reads here

$$\int_M R\theta \wedge d\theta = 2\pi \int_\Sigma R d\theta = 4\pi^2 \chi(\Sigma),$$

where  $R$  stands for the Tanaka–Webster scalar curvature, which coincides with the Riemannian scalar curvature of the base in this Seifert case; see e.g. [7]. Then, by universality of the coefficients of the heat development, the following 'fantastic factorisation' holds on *any* 3-dimensional contact manifold.

**Corollary 5.13.** *On any contact 3-manifold, the full development of  $\mathrm{Tr}_\kappa(e^{-t\Delta})$  as  $t \searrow 0$  is of type*

$$(76) \quad \mathrm{Tr}_\kappa(e^{-t\Delta}) \sim \frac{\dim V}{4\pi\sqrt{\pi t}} \int_M R\theta \wedge d\theta + \sum_{n \geq 0} t^{n/2} \int_M P_n(R, A) d\mathrm{vol},$$

where all invariant curvature polynomials  $P_n(R, A)$  factorise through Tanaka–Webster torsion  $A = \mathcal{L}_T J$  and its covariant derivatives.

In the opposite direction, when  $t \rightarrow +\infty$ , the trace formula (72) gives the asymptotic development of the dynamical theta function  $\vartheta$ . Indeed, after removing the zero eigenspace, heat decays exponentially, yielding the following property.

**Corollary 5.14.** *On CR Seifert manifolds, it holds as  $t \rightarrow +\infty$  that*

$$(77) \quad \vartheta(t) = \kappa(M, \rho) - \dim V \frac{\sqrt{\pi} \chi(\Sigma)}{\sqrt{t}} + O(e^{-Ct}).$$

This behaviour is not surprising; it can also be seen from the explicit formula (73) and Proposition 5.8, which relate the dynamical theta function to the classical Jacobi theta function, whose decay at  $+\infty$  is well known; see e.g. [46, §21.51].

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